

A STOCHASTIC SOLUTION WITH GAUSSIAN STATIONARY INCREMENTS OF THE SYMMETRIC SPACE-TIME FRACTIONAL DIFFUSION EQUATION

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Abstract

The stochastic solution with Gaussian stationary increments is established for the symmetric space-time fractional diffusion equation when $0 < \beta < \alpha \leq 2$, where $0 < \beta \leq 1$ and $0 < \alpha \leq 2$ are the fractional derivation orders in time and space, respectively. This solution is provided by imposing the identity between two probability density functions resulting (i) from a new integral representation formula of the fundamental solution of the symmetric space-time fractional diffusion equation and (ii) from the product of two independent random variables. This is an alternative method with respect to previous approaches such as the scaling limit of the continuous time random walk, the parametric subordination and the subordinated Langevin equation. A new integral representation formula for the fundamental solution of the space-time fractional diffusion equation is firstly derived. It is then shown that, in the symmetric case, a stochastic solution can be obtained by a Gaussian process with stationary increments and with a random width scale variable distributed according to an arrangement of two extremal Lévy stable densities. This stochastic solution is self-similar with stationary increments and uniquely defined in a statistical sense by the mean and the covariance structure. Numerical simulations are carried out by choosing as Gaussian process the fractional Brownian motion. Sample paths and probability density functions are shown to be in agreement with the fundamental solution of the symmetric space-time fractional diffusion equation.

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1. Introduction

Space-time fractional diffusion was originally introduced in physics by Zaslavsky [108, 110, 109] to study chaotic Hamiltonian dynamics in low dimensional systems with the specific aim to model the so-called *anomalous diffusion* (see also [113, 111, 112]). The label *anomalous diffusion* is assigned to processes whose variance does not grow linearly in time [98], in contrast to Gaussian *normal diffusion* that is mainly characterized by such linear law. Anomalous diffusion has been experimentally observed several times and definitively established in nature not only in chaotic dynamical systems (see, e.g., References [67, 93, 27, 28, 17]).

Zaslavsky argued that, since chaotic dynamics is a physical phenomenon whose evolution bridges between a completely regular integrable system and a completely random process [111], kinetic equations and statistical tools arise as modelling methods. In these cases the classical diffusion paradigm, which is based on a local and linear flux-gradient relationship, does not hold, thus a non-local and/or non-linear relationship is needed. In the Fractional Calculus approach the idea is to maintain a linear relationship, while introducing a non-local dependence in the flux-gradient relationship by means of integrals with inverse power-law kernels [89, 90]. This is in agreement with the observation of non-local behavior in many natural phenomena, such as the emergence of spatially extended coherent structures in the turbulent atmospheric boundary layer [89, 85, 86, 87].

Fractional Calculus [92, 7] is nowadays recognized to be a useful mathematical tool for modelling such linear non-local effects. In this framework, non-locality can be considered in time (*time-fractional diffusion*) [56, 18, 99, 44, 106] or in space (*space-fractional diffusion*) [18, 33, 48, 38], as well as both in space and time (*space-time fractional diffusion*) [112, 33, 95, 96, 57, 42, 43, 8, 50, 68, 64, 22, 24, 23, 6, 29].

Moreover, when there is no separation of time scales between the microscopic and the macroscopic level of the process, the randomness of the microscopic level is, at least partially, transmitted to the macroscopic level and the macroscopic dynamics is correctly described by means of non-local operators with self-similar features, hence the emergence of Fractional Calculus [46]. Further, fractional integro-differential equations are related to the fractal properties of phenomena [94]. However, the fractional kinetics strongly differs from the usual kinetics because some moments of the probability density function (PDF) of particle displacement can be infinite and the fluctuations from the equilibrium state have a broad distribution of relaxation times [101, 111, 51].

The *space-time fractional diffusion* equation reads [57]

$${}_tD_*^\beta u(x; t) = {}_xD_\theta^\alpha u(x; t), \quad -\infty < x < +\infty, \quad t \geq 0, \quad (1.1a)$$

$$u(x; 0) = u_0(x), \quad u(\pm\infty; t) = 0. \quad (1.1b)$$

where ${}_tD_*^\beta$ is the *Caputo time-fractional derivative* of order β and ${}_xD_\theta^\alpha$ is the *Riesz-Feller space-fractional derivative* of order α and asymmetry parameter θ . The real parameters α , θ and β are restricted as follows

$$0 < \alpha \leq 2, \quad |\theta| \leq \min\{\alpha, 2 - \alpha\}, \quad 0 < \beta \leq 1 \quad \text{or} \quad 1 < \beta \leq \alpha \leq 2. \quad (1.2)$$

The definitions of the fractional differential operators ${}_tD_*^\beta$ and ${}_xD_\theta^\alpha$ with a brief review of (1.1a) are reported in Appendix (see also References [57, 41]).

The general solution of (1.1a) can be represented as

$$u(x; t) = \int_{-\infty}^{+\infty} K_{\alpha, \beta}^\theta(x - \xi; t) u_0(\xi) d\xi, \quad (1.3)$$

where $K_{\alpha, \beta}^\theta(x; t)$ is the fundamental solution, or Green function, which is obtained by setting in (1.1b) the initial condition $u_0(x) = \delta(x)$. Particular cases of Eq. (1.1a) are the space-fractional diffusion equation when $\beta = 1$, the time-fractional diffusion equation when $\alpha = 2$ and the (Gaussian) parabolic diffusion equation when $\alpha = 2$ and $\beta = 1$. Furthermore when $\alpha = \beta = 2$ the D'Alembert wave equation is recovered.

Space-time fractional diffusion equation (1.1a) was analytically considered by many authors [95, 103, 37, 79, 67, 57]. The fundamental solution has been expressed by the Mellin–Barnes integral representation (A.8) [79, 57] as well as in terms of H-Fox function [79, 63].

Solutions of equation (1.1a) have been recognized to be good fitting in anomalous diffusion processes such as, for example, non-diffusive chaotic transport by Rossby waves in zonal flow [23], transport in pressure-gradient-driven plasma turbulence [23, 24, 22], transport with perturbative effects in magnetically confined fusion plasmas [25], non-diffusive tracer transport in a zonal flow under the effects of finite Larmor radius [47] or transport in point vortex flow [53].

On the physical ground, the time-fractional derivative is related to the non-Markovianity, thus long-range memory, and the space-fractional derivative to non-Gaussian particle displacement PDF with heavy tails, thus non-locality. In particular, in the Rossby waves problem, the trapping effect of the vortices gives rise to non-Markovian effects, determining the emergence of subdiffusive behavior, and the zonal shear flows give rise to non-Gaussian particle displacement [23]. In the plasma physics problem, the non-Markovian effects are due to the trapping in electrostatic eddies

and the non-Gaussian particle displacements result from avalanche-like radial relaxation events [22, 24, 23]. In the context of flows in porous media, fractional time derivatives describes particles that remain motionless for extended periods of time while fractional space derivatives model large motions through highly conductive layers or fractures [64, 9].

Castiglione *et al.* [15] have highlighted that fractional diffusion equation (1.1a) fails to model *strong anomalous diffusion*, namely those diffusive processes with the scaling laws of statistical moments depending on the order of the moment under consideration: $\langle X^m \rangle(t) \sim t^m g(m)$, where $g(m)$ is a function of the moment order m . This is typically associated with multi-scaling or multi-fractal signals. However, in many systems the fractional diffusion approximation was shown to be still valid, and this is even more true when considering the long-time behavior [98, 8].

Since (1.1a) can be understood as a Master equation and its solution as a PDF, the formulation of the underlying *stochastic process* is important to physically depict the anomalous diffusion at the micro/mesoscopic scale. In this respect, it is well-known that the classical Gaussian diffusion, namely the *Brownian motion* (Bm), is stochastically described by the Wiener process. In this paper a *stochastic solution* for the *symmetric space-time fractional diffusion* equation is derived, where stochastic solution means a stochastic process whose one-point one-time PDF is the solution of the Master (diffusion) equation. The symmetric space-time fractional diffusion equation occurs when $\theta = 0$:

$${}_t D_*^\beta u(x; t) = {}_x D_0^\alpha u(x; t) = \frac{\partial^\alpha u}{\partial |x|^\alpha}, \quad -\infty < x < +\infty, \quad t \geq 0, \quad (1.4)$$

with α and β ranging as in (1.2). Preliminary analytic solutions of the symmetric case (1.4) were computed by Saichev & Zaslavsky [95] and Gorenflo, Iskenderov & Luchko [37].

It is worth noting that the solution of (1.4) with the proper initial and boundary conditions is unique, but this uniqueness is not met in general in the underlying stochastic process. In fact, the Master equation determines only the one-point one-time PDF, thus there is an infinite number of stochastic solutions which define n -points n -times PDFs, each one characterized by different space-time correlation properties, that share the same one-point one-time PDF.

In this respect it is reminded that a stochastic solution of the Master equation (1.4) is, for example, the Continuous Time Random Walk (CTRW) [69, 70, 105]. The CTRW is a random walk model with crucial events occurring randomly in time, thus characterized not only by the probability distribution of the jumps, but also by the probability distribution of

the inter-event times. In general, the events can be correlated and so the inter-event times. However, the most frequent and reasonable assumption is that of statistical independence among inter-event times, thus defining a renewal point process [19]. Interestingly, the renewal property, which has been verified in many real phenomena [85, 87, 11, 84, 3], is associated with the emergence of metastable self-organized states, or coherent structures, and, in more detail, with an intermittent birth-death process of self-organization [87, 86, 88, 2, 1, 91, 83]. Thus, even if a rigorous derivation of fractional operators from intermittent self-organization does not exist, fractional diffusion equations are expected to emerge in these cases.

The CTRW is an exact stochastic solution when the cumulative distribution of the inter-event times is a Mittag–Leffler function [97, 34, 36]. In general, CTRWs with proper scaling limits in the inter-event time and/or jump distribution satisfy the space and/or time fractional diffusion equation, but only in the long-time limit [39, 40, 35]. Other stochastic solutions were derived by means of the parametric subordination [39, 40, 41] and of the subordinated Langevin equation [33, 106, 55]. However, all these methods, which are also interconnected, do not have stationary increments.

Moreover, we highlight here also that in certain cases the knowledge of the one-point one-time PDF is not enough to uniquely infer information on the system, as for example the information on the first passage time when geometric constraints are applied [102, 66, 100].

In this paper we propose an approach that leads to a unique stochastic solution in the sense that it is fully statistically characterised and has Gaussian stationary increments. This stochastic process is essentially associated with a *Gaussian stochastic process* and then fully characterized by its first and second moments. In particular we define and characterise a process whose one-point one-time PDF is the solution of the symmetric space-time fractional diffusion equation (1.4).

Such process is defined as the product of a Gaussian process by an independent and constant non-negative random variable distributed according to a combination of Lévy stable densities. The chosen Gaussian process may have an arbitrary temporal correlation, that however must be in agreement with the time dependent variance of the resulting final process.

The choice of this Gaussian process can be made to meet some physical constraints, as for example the geometrical constraints mentioned above [102, 66, 100], and after this some conditions for the statistical characterisation can be derived.

Here, the *fractional Brownian motion* (fBm) is chosen, among the infinite different Gaussian processes that can be used, because this choice

provides a self-similar process with Gaussian stationary increments, beside the fact that it is a simple approach for trajectory simulations.

To achieve this stochastic solution with stationary increments we firstly derive a new *integral representation formula* for the fundamental solution of the asymmetric space-time fractional diffusion equation (1.1a). Then, we consider the symmetric case corresponding to Eq. (1.4) and we exploit the identity between the PDFs resulting, on one side, from such new integral representation formula and, on the other side, from the product of two independent random variables. Finally we establish the correspondence of such product of two independent random variables with the product of a Gaussian process, e.g., the fBm, by an appropriate random variable.

The rest of the paper is organized as follows. In Section 2 we derive a new integral representation formula for the fundamental solution of the space-time fractional diffusion equation (1.1a). In Section 3 we obtain the stochastic solution of (1.4) by using the identity between the resulting PDF of the integral representation formula and the PDF of the product of two independent variables. In Section 4 we show the results of numerical simulations and, finally, we draw some discussions and conclusions in Section 5. In Appendix we summarize the main properties of space-time fractional diffusion equation (1.1a).

2. Integral representation formulae for the fundamental solution of the space-time fractional diffusion equation

2.1. Brief review on integral representation formulae. Let $Y(\tau)$, $\tau > 0$, be a stochastic process. If the parameter τ is randomized according to a second stochastic process T with non-negative increments, i.e., $\tau = T(t)$, then the resulting process $X(t) = Y(T(t))$ is said to be *subordinated* to $Y(\tau)$, which is called the *parent process*, and to be directed by $T(t)$, which is the *directing process* [32]. In diffusive processes, the parameter τ is a time-like variable and it is referred to as the *operational time* [30]. In terms of PDFs, such subordination process is embodied by the following integral formula

$$p(x; t) = \int_0^\infty \psi(x; \tau) \varphi(\tau; t) d\tau, \quad (2.1)$$

where $p(x; t)$ is the PDF of the resulting process $X(t)$, $\psi(x; \tau)$ is the PDF of the parent process $Y(\tau)$ and $\varphi(\tau; t)$ the PDF of the directing process $T(t)$.

Formula (2.1) is a particular case of *integral representation formula* that can be studied in the framework of the Mellin transform theory and interpreted as a convolution integral [61, 62]. Many integral representation

formulae, which are summarized in the following, were studied independently from their stochastic interpretation, which is not given only by the subordination approach.

The PDFs under consideration display a self-similar or scaling property, i.e., they can be reduced to functions of a similarity variable namely $F(x; t) = 1/t^\Lambda F_0(x/t^\Lambda)$, where Λ is the scaling parameter. In order to simplify the mathematical notation, and without loss of generality, hereinafter the same symbol is used for both the two-variable function $F(x; t)$ and the one-variable function $F_0(z)$, $z = x/t^\Lambda$, so that the above relationship turns out to be written as $F(x; t) = 1/t^\Lambda F(x/t^\Lambda)$. The chosen notation is not ambiguous as the meaning clearly follows from the number of independent variables.

The following valuable integral representation formula for $K_{\alpha,\beta}^\theta(x; t)$ was derived by Uchaikin & Zolotarev [104, 103]

$$K_{\alpha,\beta}^\theta(x; t) = \int_0^\infty L_\alpha^\theta(x; (t/y)^\beta) L_\beta^{-\beta}(y) dy, \tag{2.2}$$

where $L_\alpha^\theta(x; t)$ is the Lévy stable distribution with scaling parameter α and asymmetry parameter θ . In particular, the Lévy stable distribution is the fundamental solution of the space-fractional diffusion equation that is obtained as special case of (1.1a) when $\beta = 1$, i.e., $L_\alpha^\theta(x; t) = K_{\alpha,1}^\theta(x; t)$ (see Appendix). By putting $t/y = \xi^{1/\beta}$, formula (2.2) becomes [57]

$$K_{\alpha,\beta}^\theta(x; t) = \int_0^\infty L_\alpha^\theta(x; \xi) L_\beta^{-\beta}(t; \xi) \frac{t}{\beta \xi} d\xi. \tag{2.3}$$

This formula is also used in the elegant *parametric subordination* approach, developed by Gorenflo and co-authors [45, 39, 40, 41], and it is based on a systematic and consequent application of the CTRW integral equation to the various processes involved. This approach considers the parent process $Y(\tau)$ and the random walk $t = T^{-1}(\tau)$, which is the inverse of $\tau = T(t)$ and it is called the *leading process*. Hence, after the identification of the particle trajectory with the parent process, from the system composed by $X = Y(\tau)$ and $t = t(\tau)$ the dummy variable τ can be eliminated and the evolution of $X(t)$ obtained. This is similar to the set of subordinated Langevin equations proposed by Fogedby [33]:

$$\frac{dX}{d\tau} = \eta(\tau), \quad \frac{dt}{d\tau} = \xi(\tau),$$

where $\eta(\tau)$ and $\xi(\tau)$ are independent noises whose distributions are related to the parent and the leading process, respectively (see also References [52, 45, 106, 55, 31, 30]).

Integral representation formulae (2.2) and (2.3) were used also in a theoretical statistical approach to study space-time fractional diffusion [4, 6, 5, 65] as well as to derive the stochastic solution of (1.1a) [4, 64].

Further important integral representation formulae were derived in literature [57, 61]. In particular, starting from [57, Eq. (6.16)], it results:

$$K_{\alpha,\beta}^\theta(z) = \alpha \int_0^\infty \xi^{\alpha-1} M_\beta(\xi^\alpha) L_\alpha^\theta(z/\xi) \frac{d\xi}{\xi}, \quad 0 < \beta \leq 1, \quad (2.4a)$$

$$K_{\alpha,\beta}^\theta(z) = \int_0^\infty M_{\beta/\alpha}(\xi) N_\alpha^\theta(z/\xi) \frac{d\xi}{\xi}, \quad 0 < \beta/\alpha \leq 1, \quad (2.4b)$$

where $M_\nu(\xi)$, $0 < \nu < 1$, is the M-Wright/Mainardi function which is also related to the fundamental solution of the time-fractional diffusion equation that is obtained as special case of (1.1a) when $\alpha = 2$, i.e., $M_{\beta/2}(x;t)/2 = K_{2,\beta}^0(x;t)$, and $N_\alpha^\theta(x;t)$ is the fundamental solution of the neutral fractional diffusion equation that is obtained as special case of (1.1a) when $0 < \alpha = \beta < 2$, i.e., $N_\alpha^\theta(x;t) = K_{\alpha,\alpha}^\theta(x;t)$ (see Appendix for more information on M_β , L_α^θ and N_α^θ).

By replacing z with $x/t^{\beta/\alpha}$, after the changes of variable $\xi = \tau^{1/\alpha}/t^{\beta/\alpha}$ and $\xi = \tau/t^{\beta/\alpha}$ in (2.4a) and (2.4b), respectively, it follows [57, 61]:

$$t^{-\beta/\alpha} K_{\alpha,\beta}^\theta\left(\frac{x}{t^{\beta/\alpha}}\right) = \int_0^\infty L_\alpha^\theta\left(\frac{x}{\tau^{1/\alpha}}\right) t^{-\beta} M_\beta\left(\frac{\tau}{\beta}\right) \frac{d\tau}{\tau^{1/\alpha}}, \quad 0 < \beta \leq 1, \quad (2.5a)$$

$$t^{-\beta/\alpha} K_{\alpha,\beta}^\theta\left(\frac{x}{t^{\beta/\alpha}}\right) = \int_0^\infty t^{-\beta/\alpha} M_{\beta/\alpha}\left(\frac{\tau}{t^{\beta/\alpha}}\right) N_\alpha^\theta\left(\frac{x}{\tau}\right) \frac{d\tau}{\tau}, \quad 0 < \beta/\alpha \leq 1, \quad (2.5b)$$

or, analogously [57, 61]:

$$K_{\alpha,\beta}^\theta(x;t) = \int_0^\infty L_\alpha^\theta(x;\tau) M_\beta(\tau;t) d\tau, \quad 0 < \beta \leq 1, \quad (2.6a)$$

$$K_{\alpha,\beta}^\theta(x;t) = \int_0^\infty N_\alpha^\theta(x;\tau) M_{\beta/\alpha}(\tau;t) d\tau, \quad 0 < \beta/\alpha \leq 1. \quad (2.6b)$$

The symmetric case, i.e. $\theta = 0$, of formula (2.6a) was previously derived by Saichev & Zaslavsky [95]. Moreover, combining (2.3) and (2.6a) it follows the identity [57]

$$L_\beta^{-\beta}(t;\tau) \frac{t}{\beta\tau} = M_\beta(\tau;t), \quad 0 < \beta \leq 1, \quad \tau \geq 0, \quad t \geq 0, \quad (2.7)$$

and by using self-similarity properties

$$L_\beta^{-\beta}\left(\frac{t}{\tau^{1/\beta}}\right) \frac{t}{\beta\tau^{1/\beta+1}} = \frac{1}{t^\beta} M_\beta\left(\frac{\tau}{t^\beta}\right), \quad 0 < \beta \leq 1, \quad \tau \geq 0, \quad t \geq 0. \quad (2.8)$$

Since the relationship among PDFs $K_{\alpha,\beta}^\theta$, L_α^θ , M_ν and N_α^θ (see Appendix), integral representation formulae (2.6a) and (2.6b) can be re-stated also in terms of only the particle PDF $K_{\alpha,\beta}^\theta(x;t)$ with the opportune choice of parameters [57, 61], i.e.

$$K_{\alpha,\beta}^\theta(x;t) = 2 \int_0^\infty K_{\alpha,1}^\theta(x;\tau) K_{2,2\beta}^0(\tau;t) d\tau, \quad 0 < \beta \leq 1, \quad (2.9a)$$

$$K_{\alpha,\beta}^\theta(x;t) = 2 \int_0^\infty K_{\alpha,\alpha}^\theta(x;\tau) K_{2,2\beta/\alpha}^0(\tau;t) d\tau, \quad 0 < \beta/\alpha \leq 1. \quad (2.9b)$$

Formula (2.6a), or the analog ones, shows that the solution of the space-time fractional diffusion equation (1.1a) can be expressed in terms of the solution of the space-fractional diffusion equation of order α , i.e. $K_{\alpha,1}^\theta(x;t) = L_\alpha^\theta(x;t)$, and of the solution of the time-fractional diffusion equation of order 2β , i.e. $K_{2,2\beta}^0(\tau;t) = M_\beta(\tau;t)/2$, $\tau \geq 0$. Moreover, formulae (2.6a) and (2.6b), or the analog ones, by involving non-negative functions allow for interpreting $K_{\alpha,\beta}^\theta(x;t)$ as a PDF. Furthermore, it is worth noting to remark that formula (2.6b), or the analog ones, is fundamental to extend such probability interpretation to the range $1 < \beta \leq \alpha \leq 2$.

2.2. A new integral representation formula. By using formula (2.6a) a new integral representation formula for the space-time fractional diffusion can be derived. A preliminary derivation was presented in the proceedings papers [78, 76].

THEOREM 2.1. *Let $K_{\alpha,\beta}^\theta(x;t)$ be the fundamental solution of the space-time fractional diffusion equation (1.1a) with initial and boundary conditions $u(x;0) = \delta(x)$ and $u(\pm\infty;t) = 0$ and parameters α , θ , β such that*

$$0 < \alpha \leq 2, \quad |\theta| \leq \min\{\alpha, 2 - \alpha\}, \quad 0 < \beta \leq 1,$$

then the following integral representation formula holds true for $0 < x < \infty$:

$$K_{\alpha,\beta}^\theta(x;t) = \int_0^\infty L_\eta^\gamma(x;\xi) K_{\nu,\beta}^{-\nu}(\xi;t) d\xi, \quad \text{with } \alpha = \eta\nu, \quad \theta = \gamma\nu, \quad (2.10)$$

and

$$0 < \eta \leq 2, \quad |\gamma| \leq \min\{\eta, 2 - \eta\}, \quad 0 < \nu \leq 1.$$

P r o o f. It is known that the following integral representation formula for Lévy stable density holds [32, 61, 62]:

$$L_\alpha^\theta(x;t) = \int_0^\infty L_\eta^\gamma(x;\xi) L_\nu^{-\nu}(\xi;t) d\xi, \quad \alpha = \eta\nu, \quad \theta = \gamma\nu, \quad (2.11)$$

where

$$0 < \alpha \leq 2, \quad |\theta| \leq \min\{\alpha, 2 - \alpha\}, \quad (2.12a)$$

$$0 < \eta \leq 2, \quad |\gamma| \leq \min\{\eta, 2 - \eta\}, \quad 0 < \nu \leq 1. \quad (2.12b)$$

Hence, inserting (2.11) into formula (2.6a) gives:

$$\begin{aligned} K_{\alpha,\beta}^\theta(x;t) &= \int_0^\infty \left\{ \int_0^\infty L_\eta^\gamma(x;\xi) L_\nu^{-\nu}(\xi;\tau) d\xi \right\} M_\beta(\tau;t) d\tau, \\ &= \int_0^\infty L_\eta^\gamma(x,\xi) \left\{ \int_0^\infty L_\nu^{-\nu}(\xi;\tau) M_\beta(\tau;t) d\tau \right\} d\xi, \end{aligned} \quad (2.13)$$

where the exchange of integration is allowed by the fact that the involved functions are normalized PDFs. Using again (2.6a) to compute the integral into braces in (2.13), since $0 < \beta \leq 1$, integral representation formula (2.10) is obtained. \square

COROLLARY 2.1. *In the particular case $\eta = 2$ and $\gamma = 0$, so that $\nu = \alpha/2$ and $\theta = 0$, the spatial variable x emerges to be distributed according to a Gaussian PDF and integral representation formula (2.10) becomes*

$$K_{\alpha,\beta}^0(x;t) = \int_0^\infty \mathcal{G}(x;\xi) K_{\alpha/2,\beta}^{-\alpha/2}(\xi;t) d\xi, \quad 0 < \alpha \leq 2, \quad 0 < \beta \leq 1. \quad (2.14)$$

P r o o f. The proof straightforwardly follows from the fact that identities in (A.12) hold

$$L_2^0(x;t) = \mathcal{G}(x;t) = \frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}}. \quad (2.15)$$

\square

Formula (2.14) involves the Gaussian PDF $G(x,t)$ and can be stochastically interpreted in different ways, including that of a subordination process. However, in the next section we will follow an alternative interpretation, which allows for deriving a new self-similar Gaussian-based stochastic solution of the symmetric space-time fractional diffusion equation (1.4) with Gaussian stationary increments.

3. The stochastic solution with Gaussian stationary increments of the symmetric space-time fractional diffusion equation

In this Section we propose a novel approach to obtain a new stochastic solution $X_{\alpha,\beta}(t)$ of the symmetric space-time fractional diffusion equation (1.4) which has the valuable property to have stationary increments and to be fully characterized. This method is based on the correspondence of

the PDFs resulting from the integral representation formula (2.14) with the PDFs resulting from the product of two independent random variables.

This approach was preliminarily presented in the proceedings paper [76].

In general, the one-point one-time PDF is not sufficient to characterise a stochastic process. In fact, starting from a Cauchy problem whose solution can be interpreted as a PDF, there is an infinity of stochastic processes that share the same one-dimensional distribution. But, in the present approach, the correspondence with the product of an appropriate random variable and a Gaussian stochastic process, suggests a method to solve this indeterminacy. In particular, because the Gaussian process is fully characterised.

It is well known that the PDF of the product of two independent random variables is given by an integral formula [32, 61, 62]. The correspondence between a parent-directing subordination and the product of two independent random variables for self-similar processes was also highlighted in Reference [81]. This relationship is useful to establish classes of *Hurst Self-Similar with Stationary Increments* (H-SSSI) stochastic processes to model anomalous diffusion [81]. In this respect the following definition is reminded:

DEFINITION 3.1 (H-SSSI processes). A stochastic process $W(t)$, $t \geq 0$, with values in R , is a H-SSSI process if: (i) it is a self-similar process, i.e.: $W(at)$ and $a^H W(t)$ have the same finite-dimensional distributions for all $a > 0$, where H is the Hurst exponent; (ii) it has stationary increments, i.e., the distribution of the increments $W(t + \tau) - W(t)$ is invariant under the time shift transformation: $t \rightarrow t + s$.

First, the following lemma is given.

LEMMA 3.1. Let Z_1 and Z_2 be two real independent random variables whose PDFs are $p_1(z_1)$ and $p_2(z_2)$, respectively, with $z_1 \in R$ and $z_2 \in R^+$. Let Z be the random variable obtained by the product of Z_1 and Z_2^γ , i.e.:

$$Z = Z_1 Z_2^\gamma. \quad (3.1)$$

Then, denoting with $p(z)$ the PDF of Z , it results:

$$p(z) = \int_0^\infty p_1\left(\frac{z}{\lambda^\gamma}\right) p_2(\lambda) \frac{d\lambda}{\lambda^\gamma}. \quad (3.2)$$

P r o o f. The joint PDF of Z_1 and Z_2 is $p(z_1, z_2) = p_1(z_1)p_2(z_2)$ and their joint probability to be in the intervals $z_1 < Z_1 < z_1 + dz_1$ and

$z_2 < Z_2 < z_2 + dz_2$ is given by $p(z_1, z_2)dz_1dz_2$, where dz_1dz_2 is an infinitesimally small area. Relationship (3.1) suggests the following variable transformation:

$$z_1 = z/\lambda^\gamma; \quad z_2 = \lambda.$$

After the substitution of the above change of variables into the formula of the probability, it follows:

$$p(z_1, z_2) dz_1 dz_2 = p_1(z/\lambda^\gamma) p_2(\lambda) J dz d\lambda,$$

where $J = 1/\lambda^\gamma$ is the Jacobian of the transformation. Finally, integration in $d\lambda$ gives formula (3.2). \square

It is here reminded that there is a formal correspondence between the PDF of the product of two variables as stated in formula (3.2) and the integral representation formula (2.14), such that it is possible to link the mathematical expressions of the two PDFs.

By applying the changes of variables $z = xt^{-\gamma\omega}$ and $\lambda = \tau t^{-\omega}$, integral representation (2.1) is recovered from (3.2) by setting

$$\frac{1}{t^{\gamma\omega}} p\left(\frac{x}{t^{\gamma\omega}}\right) \equiv p(x; t), \quad \frac{1}{\tau^\gamma} p_1\left(\frac{x}{\tau^\gamma}\right) \equiv \psi(x; \tau), \quad \frac{1}{t^\omega} p_2\left(\frac{\tau}{t^\omega}\right) \equiv \varphi(\tau; t). \quad (3.3)$$

Then, identifying functions and parameters in (3.3) as follows

$$p(z) \equiv K_{\alpha, \beta}^0(z), \quad p_1(z_1) \equiv \mathcal{G}(z_1), \quad p_2(z_2) \equiv K_{\alpha/2, \beta}^{-\alpha/2}(z_2), \quad (3.4a)$$

$$\gamma = 1/2, \quad \omega = 2\beta/\alpha, \quad \gamma\omega = \beta/\alpha, \quad (3.4b)$$

formula (3.2) reduces to the new integral representation formula (2.14) for the symmetric space-time fractional diffusion equation.

The correspondence between the two above mechanisms, namely formulae (2.1) and (3.2), can be understood as follows [81]. Let $W(t)$, $t \geq 0$, a H-SSSI process. Hence, if parameter a is turned into a random variable, in the parent-directing subordination approach, the resulting process emerges to be $X(t) = Y(T(t)) = W(at)$ where it holds $Y(\tau) = W(\tau)$ and $\tau = T(t) = at$, and, in the approach based on the product of two independent random variables, the resulting process is $Z(t) = Z_2^\gamma Z_1(t) = a^H W(t)$ where it holds $Z_2^\gamma = a^H$ and $Z_1(t) = W(t)$. Due to the self-similarity nature of $W(t)$, processes $X(t) = W(at)$ and $Z(t) = a^H W(t)$ have the same finite-dimensional distributions. This means that the process $Z(t)$ has the same single-point single-time density of a subordinated stochastic process where the parent process $Y(\tau)$ is a self-similar process, i.e. $Y(\tau) = W(\tau)$, and the operational time τ is a line with stochastic slope, i.e. $\tau = T(t) = at$.

In terms of random variables it follows that

$$Z = X t^{-\beta/\alpha} \quad \text{and} \quad Z = Z_1 Z_2^{1/2}, \quad (3.5)$$

hence it holds

$$X = Z t^{\beta/\alpha} = Z_1 t^{\beta/\alpha} Z_2^{1/2} = G_{2\beta/\alpha}(t) \sqrt{\Lambda_{\alpha/2,\beta}}. \quad (3.6)$$

Since $p_1(z_1) \equiv \mathcal{G}(z_1)$, Z_1 is a Gaussian random variable (see (3.4a)). Consequently, the stochastic process $Z_1 t^{\beta/\alpha} = G_{2\beta/\alpha}(t)$ is a Gaussian process displaying anomalous diffusion. Further, the random variable $Z_2 = \Lambda_{\alpha/2,\beta}$ is distributed according to $p_2(z_2) \equiv K_{\alpha/2,\beta}^{-\alpha/2}(z_2)$ (see (3.4a)).

The above reasoning is based on the same constructive approach adopted by Mura [71] to built up the *generalized grey Brownian motion* [71, 73, 72, 21, 20]. In summary, we define the following class of H-SSSI processes:

DEFINITION 3.2 (Gaussian-based H-SSSI stochastic processes). Let $X_{\alpha,\beta}(t)$, $t \geq 0$, be a H-SSSI process defined by

$$X_{\alpha,\beta}(t) = \sqrt{\Lambda_{\alpha/2,\beta}} G_{2\beta/\alpha}(t), \quad 0 < \beta \leq 1, \quad 0 < \alpha \leq 2, \quad (3.7)$$

where the stochastic process $G_{2\beta/\alpha}(t)$ is a H-SSSI Gaussian process with power law variance $t^{2\beta/\alpha}$ and $\Lambda_{\alpha/2,\beta}$ is an independent constant non-negative random variable distributed according to the PDF $K_{\alpha/2,\beta}^{-\alpha/2}(\lambda)$, $\lambda \geq 0$, that is a special case of (2.6a). Then we say that $X_{\alpha,\beta}(t)$ is a Gaussian-based H-SSSI stochastic process.

The following theorem can be now proved.

THEOREM 3.1 (Stochastic solution of equation (1.4)). *The parametric class of Gaussian-based H-SSSI stochastic processes $X_{\alpha,\beta}(t)$ defined in (3.7), and depending on the parameters $0 < \beta \leq 1$ and $0 < \alpha \leq 2$, is a class of stochastic solutions of the symmetric space-time fractional diffusion equation (1.4). This means that the one-time one-point PDF of $X_{\alpha,\beta}(t)$ is the fundamental solution of equation (1.4), namely the PDF $K_{\alpha,\beta}^0(x;t)$ defined in (2.14).*

P r o o f. For the given sequence of times (t_1, \dots, t_n) , the joint PDF $f_{\alpha,\beta}$ of the n -dimensional particle random vector $\mathbf{X} = (X_{\alpha,\beta}(t_1), \dots, X_{\alpha,\beta}(t_n))$ to be in the position vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ can be derived from (3.2) and (3.4a) by using the Kolmogorov extension theorem and it results to be

$$f_{\alpha,\beta}(x_1, x_2, \dots, x_n; \gamma_{\alpha,\beta}) = \frac{1}{\sqrt{(2\pi\lambda)^n \det \gamma_{\alpha,\beta}}} \int_0^\infty \exp \left\{ -\frac{1}{2\lambda} \mathbf{x}^T [\gamma_{\alpha,\beta}]^{-1} \mathbf{x} \right\} K_{\alpha/2,\beta}^{-\alpha/2}(\lambda) d\lambda, \quad (3.8)$$

where $\gamma_{\alpha,\beta}(t_i, t_j)$ is the covariance matrix of the random vector corresponding to the Gaussian process $G_{2\beta/\alpha}(t)$ and $[\gamma_{\alpha,\beta}]^{-1}$ is the inverse covariance matrix. The one-time one-point PDF ($n = 1$) is

$$f_{\alpha,\beta}(x; t) = \int_0^\infty \frac{1}{\lambda^{1/2}} \mathcal{G}\left(\frac{x t^{-\beta/\alpha}}{\lambda^{1/2}}\right) K_{\alpha/2,\beta}^{-\alpha/2}(\lambda) d\lambda = K_{\alpha,\beta}^0(x t^{-\beta/\alpha}), \quad (3.9)$$

where the scaling relationship $x/t^{\beta/\alpha}$, which drives the anomalous diffusion scaling, has been taken into account.

Formula (3.9) is the mathematical expression of the one-time one-point PDF of the Gaussian-based H-SSSI stochastic process $X_{\alpha,\beta}(t)$ given in (3.7). After the change of variable $\lambda = \tau t^{-2\beta/\alpha}$, formula (3.9) can be rewritten in the following way:

$$\int_0^\infty \frac{1}{\tau^{1/2}} \mathcal{G}\left(\frac{x}{\tau^{1/2}}\right) K_{\alpha/2,\beta}^{-\alpha/2}\left(\frac{\tau}{t^{2\beta/\alpha}}\right) \frac{d\tau}{t^{2\beta/\alpha}} = t^{-\beta/\alpha} K_{\alpha,\beta}^0\left(\frac{x}{t^{\beta/\alpha}}\right). \quad (3.10)$$

Considering self-similarity scaling, this expression is the same given in formula (2.14), which is the integral representation formula for the solution of the symmetric space-time fractional diffusion equation (1.4).

Then, the equivalence of the two expressions (3.9) and (2.14) demonstrates that the H-SSSI Gaussian-based stochastic process $X_{\alpha,\beta}(t)$, defined in equation (3.7), is a stochastic solution of the symmetric space-time fractional diffusion equation (1.4), thus proving Theorem 3.1. \square

It is worth noting that formula (3.9) represents a superposition of Gaussian processes depending on the value of the random multiplicative factor $\sqrt{\Lambda_{\alpha/2,\beta}}$ and that the process $X_{\alpha,\beta}(t)$ is Gaussian conditional on the value of this multiplicative factor. The multiplicative factor can be understood, for example, as related to the diffusion coefficient in analogy with ideas discussed elsewhere [80, 77].

The stochastic process $X_{\alpha,\beta}(t)$ stated in (3.7) generalizes Gaussian processes, which are recovered when $\alpha = 2$ and $\beta = 1$. Similarly to Gaussian process, even this process is uniquely determined by the mean and the autocovariance structure. This property directly follows from the fact that $G_{2\beta/\alpha}(t)$ is a Gaussian stochastic process and $\Lambda_{\alpha/2,\beta}$ is an independent constant non-negative random variable.

Before to discuss the numerical simulations it is highlighted that any Gaussian process $G_{2\beta/\alpha}(t)$ can be used to build up the stochastic solution (3.7), because each Gaussian process is characterized by a particular covariance matrix. The following numerical simulations are performed by choosing the standard fBm with Hurst exponent $H = \beta/\alpha < 1$ as Gaussian

stochastic process $G_{2\beta/\alpha}(t)$. This choice constraints parameters α and β to fall inside the intervals

$$0 < \beta \leq 1, \quad 0 < \beta < \alpha \leq 2. \quad (3.11)$$

Note that after this choice the stochastic trajectories $X_{\alpha,\beta}(t)$ have Gaussian stationary increments.

4. Numerical simulations

The fBm is the natural choice of the Gaussian process $G_{2\beta/\alpha}(t)$ in the formulation of the the Gaussian-based H-SSSI stochastic process $X_{\alpha,\beta}(t)$ defined in (3.7). The foundation of $X_{\alpha,\beta}(t)$ on the fBm is a remarkable property, as it allows to reduce a large number of issues to the analysis of the fBm, which has been largely studied (see, e.g., Reference [10]). Moreover, the stationarity of increments makes the stochastic process (3.7) more efficient for the simulations of stochastic trajectories by means of Monte Carlo methods.

Now, given a set of time points (t_1, t_2, \dots, t_n) , the fBm is defined by the following covariance matrix:

$$\gamma_{\alpha,\beta}(t_i, t_j) = t_i^{2\beta/\alpha} + t_j^{2\beta/\alpha} - |t_i - t_j|^{2\beta/\alpha}, \quad i, j = 1, \dots, n. \quad (4.1)$$

In order to perform numerical simulations of the stochastic solution $X_{\alpha,\beta}(t)$ stated in equation (3.7), both the generation of the random variable $\Lambda_{\alpha/2,\beta}$, distributed according to $K_{\alpha/2,\beta}^{-\alpha/2}(\lambda)$, and the fBm $G_{2\beta/\alpha}(t)$ are here discussed.

From (2.13) and (2.10), according to (2.6a), it follows:

$$K_{\alpha/2,\beta}^{-\alpha/2}(\xi; t) = \int_0^\infty L_{\alpha/2}^{-\alpha/2}(\xi; \tau) M_\beta(\tau; t) d\tau, \quad 0 < \beta \leq 1, \quad (4.2)$$

and, using the self-similarity property:

$$t^{-2\beta/\alpha} K_{\alpha/2,\beta}^{-\alpha/2}\left(\frac{\xi}{t^{2\beta/\alpha}}\right) = \int_0^\infty L_{\alpha/2}^{-\alpha/2}\left(\frac{\xi}{\tau^{2/\alpha}}\right) M_\beta\left(\frac{\tau}{t^\beta}\right) \frac{d\tau}{\tau^{2/\alpha} t^\beta}, \quad 0 < \beta \leq 1. \quad (4.3)$$

After the changes of variable $\xi = t^{2\beta/\alpha}\lambda$ and $\tau = t^\beta y$, it holds:

$$K_{\alpha/2,\beta}^{-\alpha/2}(\lambda) = \int_0^\infty L_{\alpha/2}^{-\alpha/2}\left(\frac{\lambda}{y^{2/\alpha}}\right) M_\beta(y) \frac{dy}{y^{2/\alpha}}, \quad 0 < \beta \leq 1. \quad (4.4)$$

The representation integral (4.4) suggests that $\Lambda_{\alpha/2,\beta}$ can be computed by means of the product of two independent random variables, see Section 3, i.e.:

$$\Lambda_{\alpha/2,\beta} = \Lambda_1 \cdot \Lambda_2 = \mathcal{L}_{\alpha/2}^{ext} \cdot \mathcal{M}_\beta^{2/\alpha}, \quad (4.5)$$

where $\Lambda_1 = \mathcal{L}_{\alpha/2}^{ext}$ and $\Lambda_2 = \mathcal{M}_\beta$ are distributed according to the extremal stable density $L_{\alpha/2}^{-\alpha/2}(\lambda_1)$ and to the density $M_\beta(\lambda_2)$, respectively, so that $\lambda = \lambda_1 \lambda_2$.

Then, from (2.8) and setting $t = 1$, it follows that the random variable \mathcal{M}_β can be determined by an extremal stable random variable according to [13]

$$\mathcal{M}_\beta = [\mathcal{L}_\beta^{ext}]^{-\beta} . \quad (4.6)$$

Finally, the random variable $\Lambda_{\alpha/2,\beta}$ is computed by the product

$$\Lambda_{\alpha/2,\beta} = \mathcal{L}_{\alpha/2}^{ext} \cdot [\mathcal{L}_\beta^{ext}]^{-2\beta/\alpha} . \quad (4.7)$$

In summary, the stochastic solution (3.7) of the symmetric space-time fractional diffusion equation (1.4) is numerically simulated by the following process

$$X_{\alpha,\beta}(t) = \sqrt{\mathcal{L}_{\alpha/2}^{ext}} \cdot [\mathcal{L}_\beta^{ext}]^{-\beta/\alpha} \cdot G_{2\beta/\alpha}(t) , \quad (4.8)$$

and the random generation is discussed below.

The computer generation of extremal stable random variables of order $0 < \mu < 1$ is obtained by using the well-known method by Chambers, Mallows and Stuck [16, 107]

$$\mathcal{L}_\mu^{ext} = \frac{\sin[\mu(r_1 + \pi/2)]}{(\cos r_1)^{1/\mu}} \left\{ \frac{\cos[r_1 - \mu(r_1 + \pi/2)]}{-\ln r_2} \right\}^{(1-\mu)/\mu} , \quad 0 < \mu < 1 , \quad (4.9)$$

where r_1 and r_2 are random variables uniformly distributed in $(-\pi/2, \pi/2)$ and $(0, 1)$, respectively.

Regarding the fBm $G_{2H}(t)$, with Hurst exponent $H = \beta/\alpha$ and variance $\langle G_{2H}^2 \rangle = 2t^{2H}$, the Hosking direct method is here applied for the range $0 < H < 1$ [49, 26]. In particular, the so-called fractional Gaussian noise (fGn) is firstly generated following its definition over the set of integer numbers:

$$Y_{2H}(n) = G_{2H}(n+1) - G_{2H}(n) \Leftrightarrow G_{2H}(n+1) = G_{2H}(n) + Y_{2H}(n) . \quad (4.10)$$

Finally the fBm is generated as a sum of stationary increments, which are generated according to the following stationary auto-correlation function, defined over integer numbers ($n \geq 0$):

$$\begin{aligned} \Gamma_{x,2H}(n) &= \langle Y_{2H}(k)Y_{2H}(k+n) \rangle \\ &= \frac{1}{2} [|n-1|^{2H} - |n|^{2H} + |n+1|^{2H}] . \end{aligned} \quad (4.11)$$

By implementing the Hosking method, a set of stochastic trajectories, i.e., sample paths, according to the fBm with Hurst exponent $H = \beta/\alpha$

were generated. In order to obtain the corresponding trajectory of the stochastic process of equation (3.7), each fBm sample path was then multiplied by the multiplicative factor $\sqrt{\Lambda_{\alpha/2,\beta}}$. It is worth noting that $\Lambda_{\alpha/2,\beta}$ is not a stochastic process evolving in time, but a constant random variable characterizing the random wideness scale of the single sample path.

The numerical simulations were carried out, following the Monte Carlo approach, by means of pseudo-random generators. For a given set of parameter values (α,β) , the number of trajectories generated were 10^4 and the number of time steps 10^3 . Following equation (4.10), a unitary time step was used as a natural choice to generate the stationary increments of fBm. Changing the time scale requires changing the time step, and the associated trajectories can be simply derived without any further numerical simulations by exploiting the self-similar property.

In Figs. 1-7 the results of numerical simulations have been reported for four different values of the Hurst exponent: $H = 0.25, 0.\bar{3}, 0.5, 0.9$. In Figs. 1-4 two different couples of parameters (α,β) , but giving the same value of $H = \beta/\alpha$, are compared. Some sample paths are reported for each case (left panels) together with the corresponding PDF (right panels). In the right panels the numerical PDFs are compared with the analytical solutions of Eq. (1.1a) by means of a matching between the convergent series (for $x \rightarrow 0$) and the asymptotic expansions (for $x \rightarrow \pm\infty$) as derived in [57]. As expected, the spreading of trajectories increases as H increases, with a dramatic increase in the neighborhood of $H = 1$ (see Fig. 4). It is interesting to note that, in the process $X_{\alpha,\beta}(t)$, even if driven by the fBm $G_{2H}(t)$, the most interesting effects come from the random wideness scale $\Lambda_{\alpha/2,\beta}$, which is essentially given by the multiplication of two extremal Lévy random variables. This means that a relatively smooth behaviour in a great portion of sample paths is observed, in the sense that they show similar random wideness scale going from one sample path to another. However, a small but non-negligible subset of sample paths display very different random wideness scales, and this is in fact related to the random variables $\Lambda_{\alpha/2,\beta}$, characterizing the ensemble of sample paths. This can be seen by comparing Fig. 6, where sample paths from a simple fBm with $H = 0.9$ are reported, with the sample paths in Figs. 1-4. In fact, even if the fBm sample paths show a great spreading (Fig. 6), at the same time they show a much higher degree of uniformity of spanning over the sample paths with respect to the sample paths shown in Figs. 1-4.

Another fundamental feature of the Lévy random wideness scale is clarified by comparing Fig. 4 (top panels) and Fig. 5. Figure 5 is a zoom of Fig. 4 (top panels), where only the sample paths with small wideness scales are reported. This is caused by the Lévy-based random wideness scale $\Lambda_{\alpha/2,\beta}$

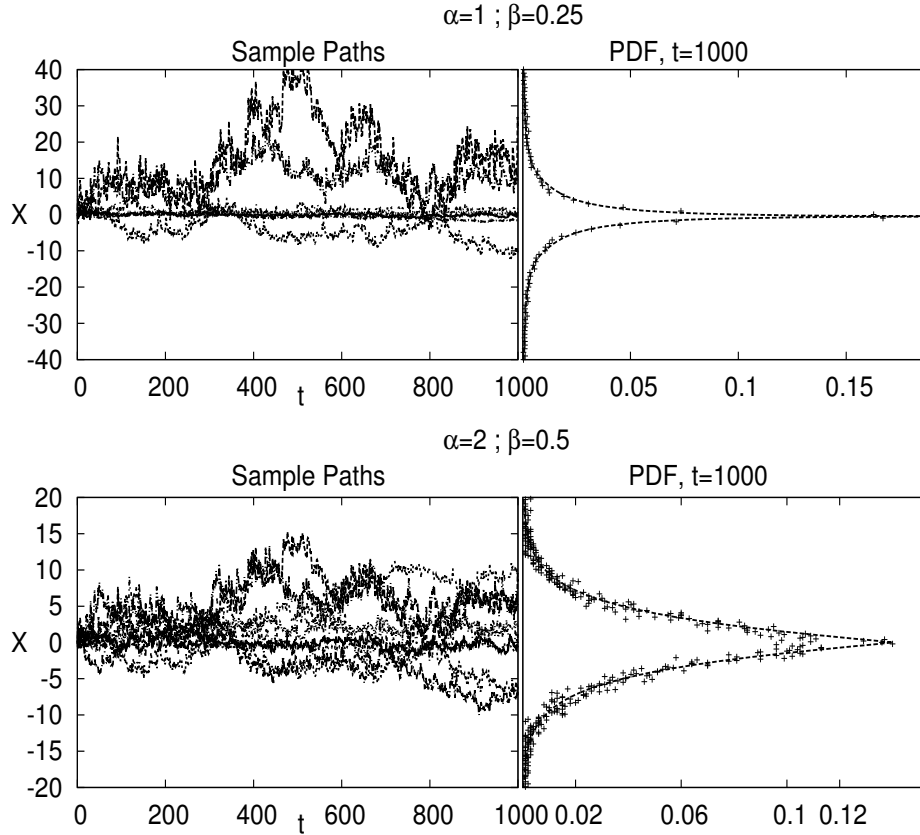


FIGURE 1. Comparison for different couples of parameters (α, β) but the same $H = \beta/\alpha$, here $H = \beta/\alpha = 0.25$. Sample paths are reported on the left panels. In the right panels the PDFs of the simulated stochastic processes $X_{\alpha,\beta}(t)$ defined in (3.7) are compared with the analytical solutions of Eq. (1.1a).

and it is strongly connected with the emergence of inverse power-law tail in the range of large $|X|$ (for each fixed time). Notice that the stochastic solution $X_{\alpha,\beta}(t)$, and the associated algorithm based on the fBm with stationary increments and a random wideness scale, well reproduces this power-law decay, as it is shown in Fig. 7 for the case of $\alpha = 0.5$.

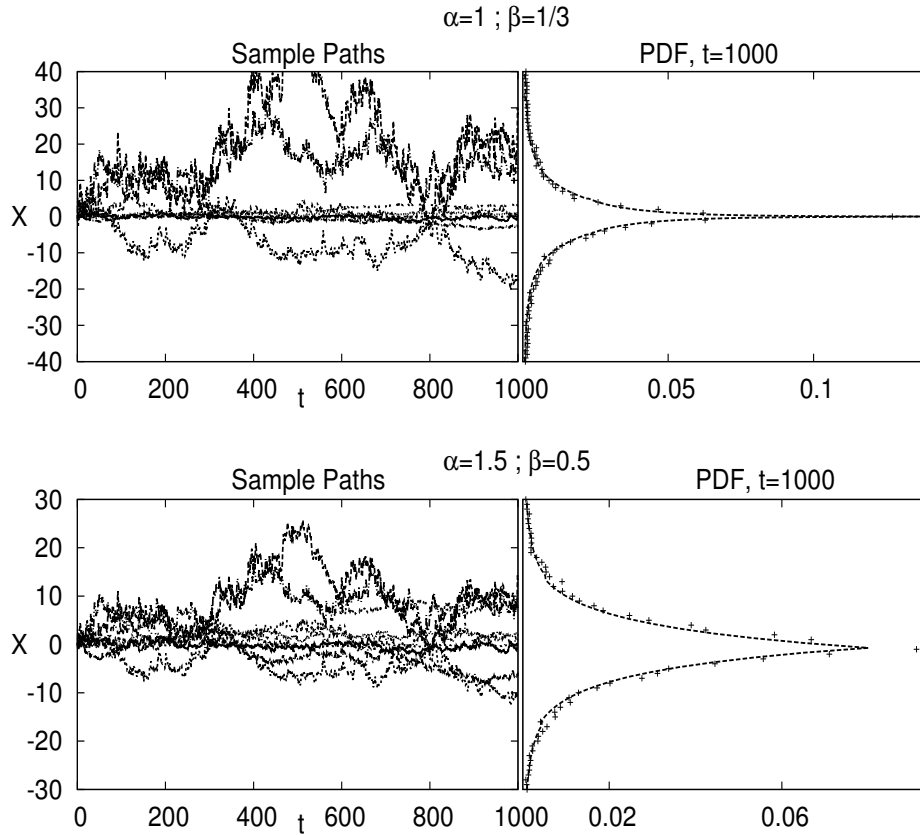


FIGURE 2. Same as Fig. 1 with $H = \beta/\alpha = 1/3$.

5. Discussion and conclusions

In order to provide a new microscopic physical insight to anomalous diffusion, in the present paper we addressed the problem of finding a Gaussian-based stochastic solution of the symmetric space-time fractional diffusion equation (1.4). The adopted method is based on the fact that the resulting PDFs from a new integral representation formula (2.10), which holds for the fundamental solution of equation (1.1a), and from the product of two independent variables are equal. The general integral representation formula (2.10) turns to be the integral formula (2.14) when considering the symmetric space-time fractional diffusion equation (1.4). This result is exploited to select a suitable independent constant and non-negative random variable and a stochastic process that are used to build up, by means of

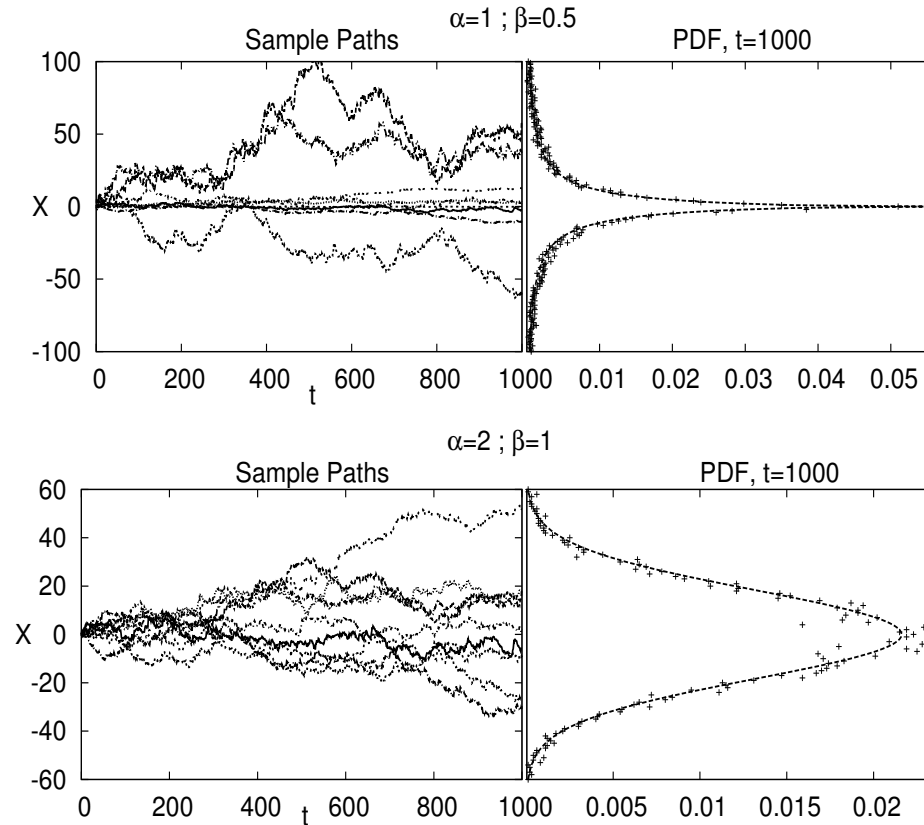
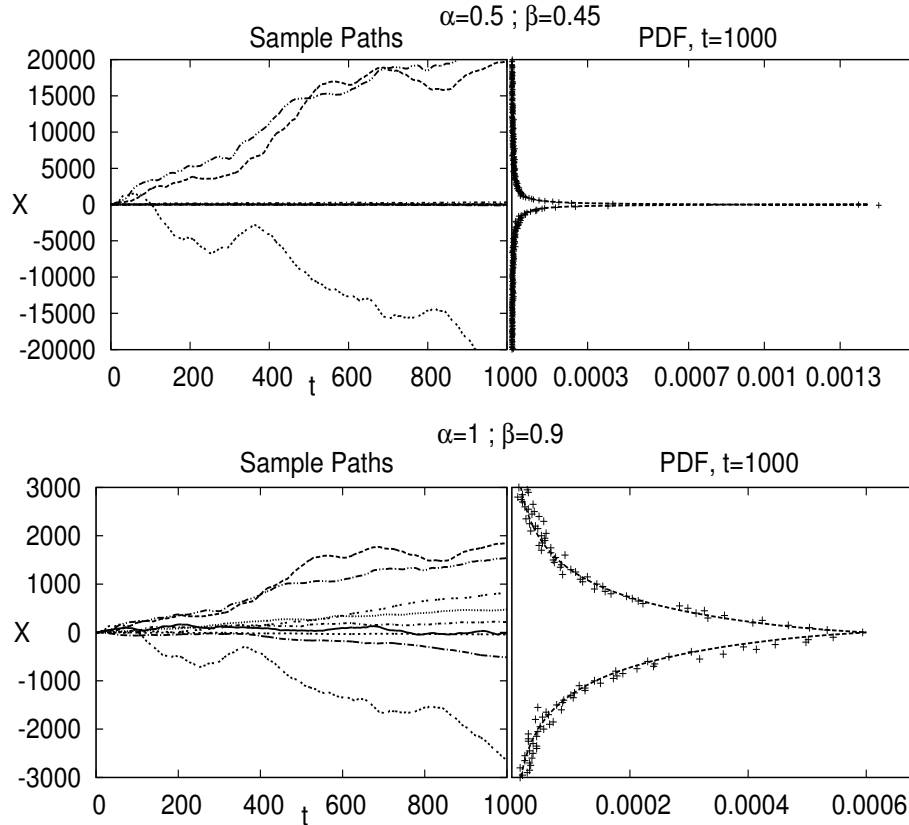


FIGURE 3. Same as Fig. 1 with $H = \beta/\alpha = 0.5$.

their product, the Gaussian-based H-SSSI stochastic process (3.7), whose one-time one-point PDF is the solution of (1.4).

The fBm has been chosen as a natural prototype for the Gaussian process driving this stochastic solution, but different choices can be made for the Gaussian part. Along this line, the stochastic process (3.7) emerges to be the product of a fBm and an independent constant and non-negative random variable distributed according to a combination of two extremal Lévy distributions.

In literature, several stochastic solutions of the symmetric space-time fractional equation (1.4) have been derived (see, e.g., References [33, 64, 97, 41]), but, up to our knowledge, the stochastic process $X_{\alpha,\beta}(t)$ here presented

FIGURE 4. Same as Fig. 1 with $H = \beta/\alpha = 0.9$.

in (3.7) is the first stochastic solution that is also a H-SSSI process based on a Gaussian process with stationary increments.

Moreover, process (3.7) shares with Gaussian processes and the generalized grey Brownian motion [73], which is the stochastic solution of the Erdélyi–Kober fractional diffusion equation [74, 80], the valuable property to be fully characterized by only the first and second moments of the Gaussian driving jumps (i.e., the temporal covariance matrix $\gamma_{\alpha,\beta}(t_i, t_j)$).

The well known characteristic behaviour of anomalous diffusive processes clearly emerge from the numerical simulations. In particular, when $\alpha \neq 2$, sample paths show very long jumps and these jumps generate the power-law decay of the PDF.

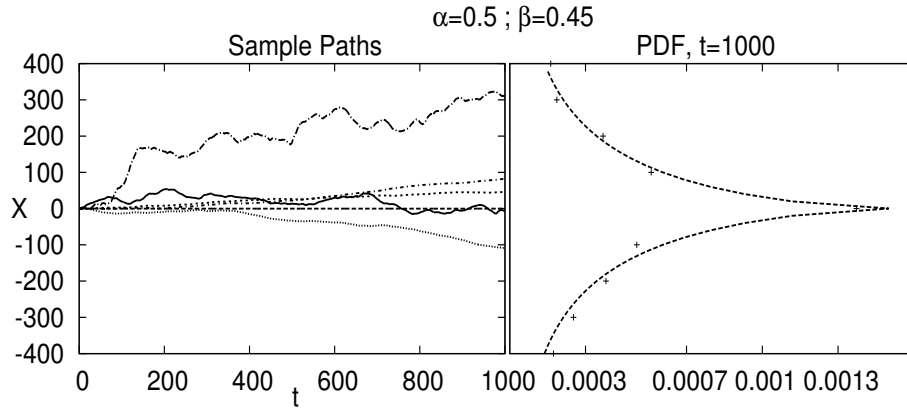


FIGURE 5. $H = \beta/\alpha = 0.9$. Zoom of the X -range (vertical axis) of the top (both left and right) panels in Fig. 4.

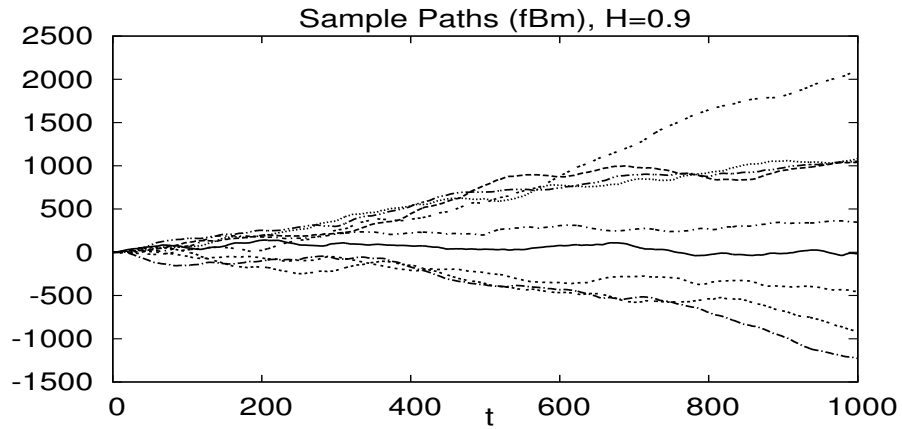


FIGURE 6. Sample paths from the fBm algorithm for $H = 0.9$, to be compared with sample paths of Fig. 4 (left panels).

Finally, valuable spatial and temporal characteristics can be obtained with the derived stochastic solution about both the diffusive process and the medium. Among these computable characteristics there are, for example, the length and time scales associated to memory decay and to trapping effects, the diffusion features according to the Taylor–Green–Kubo formula and also the occurrence of the ergodicity breaking. These features cannot

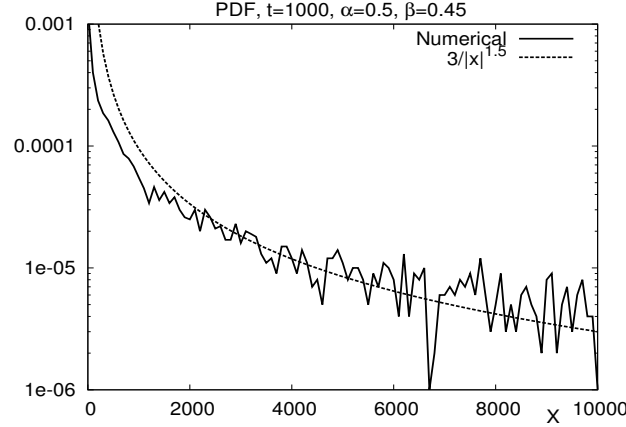


FIGURE 7. $H = \beta/\alpha = 0.9, \alpha = 0.5$. The PDF evaluated from numerical simulations well reproduces the theoretical power-law decay in the tail (large x): $\text{PDF}(x) \sim 1/|x|^{\alpha+1}$.

be acquired when only the evolution equation of the particle PDF is known and they will be investigated with the derived stochastic process in future developments of the research.

Appendix: The space-time fractional diffusion equation

Space-time fractional diffusion equation (1.1a) is obtained from the ordinary diffusion equation by replacing the first order time derivative and the second order space derivative with the *Caputo* time-fractional derivative of real order β and the *Riesz-Feller* space-fractional derivative of real order α , respectively. The *Caputo* time-fractional derivative ${}_t D_*^\beta$ is defined by its Laplace transform as

$$\int_0^{+\infty} e^{-st} \left\{ {}_t D_*^\beta u(x; t) \right\} dt = s^\beta \tilde{u}(x; s) - \sum_{n=0}^{m-1} s^{\beta-1-n} u^{(n)}(x; 0^+), \quad (\text{A.1})$$

with $m - 1 < \beta \leq m$ and $m \in \mathbb{N}$. The *Riesz-Feller* space-fractional derivative ${}_x D_\theta^\alpha$ is defined by its Fourier transform according to

$$\int_{-\infty}^{+\infty} e^{+i\kappa x} \left\{ {}_x D_\theta^\alpha u(x; t) \right\} dx = -|\kappa|^\alpha e^{i(\text{sign}\kappa)\theta\pi/2} \hat{u}(\kappa; t), \quad (\text{A.2})$$

with α and θ as in (1.2).

In literature the time-fractional derivative is sometimes considered in the Riemann–Liouville sense. The relationship of the time-fractional Riemann–Liouville derivative with the time-fractional derivative in the Caputo sense

is the following [41]

$${}_tD_*^\beta u(x; t) = {}_tD^\beta u(x; t) - \frac{t^{-\beta}}{\Gamma(1-\beta)} u(x; 0), \quad (\text{A.3})$$

and (1.1a) becomes

$${}_tD^\beta u(x; t) = {}_xD_\theta^\alpha u(x; t) + \frac{t^{-\beta}}{\Gamma(1-\beta)} u(x; 0), \quad -\infty < x < +\infty, \quad t \geq 0. \quad (\text{A.4})$$

Equation (1.1a) is stated also as

$$\frac{\partial u}{\partial t} = {}_tD^{1-\beta} [{}_xD_\theta^\alpha u(x; t)]. \quad (\text{A.5})$$

However, it is possible to show that the fundamental solutions of (1.1a), (A.4) and (A.5) are equal [41].

When $1 < \beta \leq 2$ a second initial condition is needed corresponding to $u_t(x; 0) = \left. \frac{\partial u}{\partial t} \right|_{t=0}$, and two Green functions follow according to the initial conditions $\{u(x; 0) = \delta(x), u_t(x; 0) = 0\}$ and $\{u(x; 0) = 0, u_t(x; 0) = \delta(x)\}$, respectively. However, this second Green function emerges to be a primitive (with respect to the variable t) of the first Green function, so that it cannot be interpreted as a PDF because it is no longer normalized over x [60]. Hence, solely the first Green function can be considered for diffusion problems.

In general, the fundamental solution $K_{\alpha, \beta}^\theta(x; t)$ algebraically decreases as $|x|^{-(\alpha+1)}$, thus it belongs to the domain of attraction of the Lévy stable densities of index α . Moreover, $K_{\alpha, \beta}^\theta(x; t)$ obeys the following self-similarity, or scaling, relationship:

$$K_{\alpha, \beta}^\theta(x; t) = t^{-\beta/\alpha} K_{\alpha, \beta}^\theta\left(\frac{x}{t^{\beta/\alpha}}\right), \quad (\text{A.6})$$

and it meets the following *symmetry relation*

$$K_{\alpha, \beta}^\theta(-x; t) = K_{\alpha, \beta}^{-\theta}(x; t), \quad (\text{A.7})$$

which allows for the restriction of the analysis to $x \geq 0$. When $x \geq 0$ the analytical solution of (1.1a) can be expressed by the following Mellin–Barnes integral representation [79, 57]

$$K_{\alpha, \beta}^\theta(x; t) = \frac{1}{\alpha x} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{s}{\alpha}\right) \Gamma\left(1-\frac{s}{\alpha}\right) \Gamma(1-s)}{\Gamma\left(1-\frac{\beta}{\alpha}s\right) \Gamma(\rho s) \Gamma(1-\rho s)} \left(\frac{x}{t^{\beta/\alpha}}\right)^s ds, \quad (\text{A.8})$$

where $\rho = \frac{\alpha - \theta}{2\alpha}$ and γ is a suitable real constant. Solution (A.8) can be also expressed in terms of H-Fox function [79, 63].

The special cases of space-time fractional diffusion equation (1.1a) are the following.

The *space-fractional diffusion* equation is obtained when $\{0 < \alpha < 2, \beta = 1\}$ such that

$$K_{\alpha,1}^{\theta}(x;t) = L_{\alpha}^{\theta}(x;t) = t^{-1/\alpha} L_{\alpha}^{\theta} \left(\frac{x}{t^{1/\alpha}} \right), \quad 0 < x < \infty, \quad (\text{A.9})$$

where $L_{\alpha}^{\theta}(x)$ is the class of strictly stable densities with algebraic tail decaying as $|x|^{-(\alpha+1)}$ and infinite variance, where α and θ are the scaling and asymmetry parameters, respectively. Moreover, stable PDFs with $0 < \alpha < 1$ and extremal value of the asymmetry parameter θ are one-sided with support R_0^+ if $\theta = -\alpha$ and R_0^- if $\theta = +\alpha$.

The *time-fractional diffusion* equation is obtained when $\{\alpha = 2, 0 < \beta < 2\}$ such that

$$K_{2,\beta}^0(x;t) = \frac{1}{2} M_{\beta/2}(x;t) = \frac{1}{2} t^{-\beta/2} M_{\beta/2} \left(\frac{x}{t^{\beta/2}} \right), \quad x \geq 0, \quad (\text{A.10})$$

where $M_{\nu}(x)$, $0 < \nu < 1$, is the M-Wright/Mainardi density [59, 58, 12, 13, 14, 75, 82] which has stretched exponential tails and finite variance growing in time with the power law t^{β} . Since $\alpha = 2$, according to (1.2), it holds $\theta = 0$, then the PDF is symmetric and the extension to $-\infty < x < +\infty$ is obtained by replacing x with $|x|$ in (A.10).

The *neutral fractional diffusion* equation is obtained when $\{0 < \alpha = \beta < 2\}$ whose solution can be expressed in explicit form by non-negative simple elementary functions [95, 57], i.e., when $x \geq 0$

$$K_{\alpha,\alpha}^{\theta}(x;t) = N_{\alpha}^{\theta}(x;t) = \frac{t^{-1}}{\pi} \frac{(x/t)^{\alpha-1} \sin[\frac{\pi}{2}(\alpha - \theta)]}{1 + 2(x/t)^{\alpha} \cos[\frac{\pi}{2}(\alpha - \theta)] + (x/t)^{2\alpha}}. \quad (\text{A.11})$$

Recently Luchko [54] has considered and analyzed the case $1 < \alpha < 2$ and $\theta = 0$ of (A.11). Moreover, the PDF given in equation (A.11), with $0 < \alpha < 1$, emerged in the study of finite Larmor radius effects on non-diffusive tracer transport in a zonal flow [47]. Numerical evidences of the same PDF also emerged in non-diffusive chaotic transport by Rossby waves in zonal flow [23].

The local *classical diffusion* equation is obtained when $\{\alpha = 2, \beta = 1\}$, and its Gaussian solution is recovered as a limiting case from both the space-fractional ($\alpha = 2$) and the time-fractional ($\beta = 1$) diffusion equations, i.e.,

$$K_{2,1}^0(x;t) = L_2^0(x;t) = \frac{1}{2} M_{1/2}(x;t) = \mathcal{G}(x;t) = \frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}}, \quad x \geq 0. \quad (\text{A.12})$$

The last special case is the limit case of the *D'Alembert wave equation*, $\{\alpha = 2, \beta = 2\}$, and it holds

$$K_{2,2}^0(x;t) = \frac{1}{2} M_1(x;t) = \frac{1}{2} \delta(x-t), \quad 0 < x < \infty. \quad (\text{A.13})$$

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