

Sliding mode control for a phase field system related to tumor growth

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Abstract

In the present contribution we study the sliding mode control (SMC) problem for a diffuse interface tumor growth model coupling a viscous Cahn–Hilliard type equation for the phase variable with a reaction–diffusion equation for the nutrient. First, we prove the well-posedness and some regularity results for the state system modified by the state-feedback control law. Then, we show that the chosen SMC law forces the system to reach within finite time the sliding manifold (that we chose in order that the tumor phase remains constant in time). The feedback control law is added in the Cahn–Hilliard type equation and leads the phase onto a prescribed target φ^* in finite time.

Key words: sliding mode control, Cahn–Hilliard system, reaction–diffusion equation, tumor growth, nonlinear boundary value problem, state-feedback control law.

AMS (MOS) Subject Classification: 34H15, 35K25, 35K61, 93B52, 92C50, 97M60.

1 Introduction

Sliding mode control (SMC) - which is today considered a classic instrument for regulation of continuous or discrete systems in finite-dimensional settings (see e.g. the monographs

[1, 18–20, 31, 42, 43, 46]) - has been acknowledged as one of the basic approaches to the design of robust controllers for nonlinear complex dynamics that work under uncertainty. One of the main example of complex systems studied nowadays both in biomedical and mathematical literatures is related to the tumor growth dynamics. For the case of an incipient tumor, i.e., before the development of quiescent cells, the studied diffuse interface type phase field models often consist of a Cahn–Hilliard equation coupled with a reaction-diffusion equation for the nutrient (cf., e.g., [16, 28–30]). In this work, in particular, we consider the problem of sliding mode control for a tumor growth model recently introduced in [28]. In comparison with [28], we have neglected here the effects of chemotaxis and active transport, but the new feature of (1.2) is the inclusion of the SMC law $\rho \operatorname{sign}(\varphi - \varphi^*)$, where ρ is a positive parameter that will be chosen large enough. This term forces the system trajectories onto the sliding surface $\varphi = \varphi^*$ in finite time. All in all, we consider here the following viscous Cahn–Hilliard/Reaction-Diffusion model for tumor growth

$$\partial_t \varphi - \Delta \mu = (\gamma_1 \sigma - \gamma_2) p(\varphi) \quad \text{in } Q := \Omega \times (0, T) \quad (1.1)$$

$$\mu = \tau \partial_t \varphi - \Delta \varphi + F'(\varphi) + \rho \operatorname{sign}(\varphi - \varphi^*) \quad \text{in } Q \quad (1.2)$$

$$\partial_t \sigma - \Delta \sigma = -\gamma_3 \sigma p(\varphi) + \gamma_4 (\sigma_s - \sigma) + g \quad \text{in } Q \quad (1.3)$$

where Ω is the domain in which the evolution takes place, T is some final time, φ denotes the difference in volume fraction, where $\varphi = 1$ represents the tumor phase and $\varphi = -1$ represents the healthy tissue phase, μ is the chemical potential and σ is the concentration of a nutrient for the tumor cells (e.g., oxygen or glucose). Moreover, τ is a positive viscosity coefficient, γ_i for $i = 1, \dots, 4$ denotes the positive constant proliferation rate, apoptosis rate, nutrient consumption rate, and nutrient supply rate, respectively. The term $\gamma_1 p(\varphi) \sigma$ models the proliferation of tumor cells which is proportional to the concentration of the nutrient, the term $\gamma_2 p(\varphi)$ models the apoptosis of tumor cells, and $\gamma_3 p(\varphi) \sigma$ models the consumption of the nutrient only by the tumor cells. The constant σ_s denotes the nutrient concentration in a pre-existing vasculature, and $\gamma_4 (\sigma_s - \sigma)$ models the supply of nutrient from the blood vessels if $\sigma_s > \sigma$ and the transport of nutrient away from the domain Ω if $\sigma_s < \sigma$. The function g is a source term which may represent the supply of a nutrient (see [7]), or even a drug in a chemotherapy. Moreover, F' stands for the derivative of a double-well potential F and p is a smooth nonnegative proliferation function on the domain of F . Typical examples of potentials, meaningful in view of applications, are

$$F_{reg}(r) = \frac{1}{4} (r^2 - 1)^2, \quad r \in \mathbb{R} \quad (1.4)$$

$$F_{log}(r) = (1+r) \ln(1+r) + (1-r) \ln(1-r) - c_0 r^2, \quad r \in (-1, 1) \quad (1.5)$$

$$F_{obs}(r) = I(r) - c_0 r^2, \quad r \in \mathbb{R} \quad (1.6)$$

where $c_0 > 1$ in (1.5) in order to produce a double-well, while c_0 is an arbitrary positive number in (1.6), and the function I in (1.6) is the indicator function of $[-1, 1]$, i.e., it takes the values 0 or $+\infty$ according to whether or not r belongs to $[-1, 1]$. The potentials (1.4) and (1.5) are the classical regular potential and the so-called logarithmic potential, respectively. More generally, the potential F could be just the sum $F = \widehat{\beta} + \widehat{\pi}$, where $\widehat{\beta}$ is a convex function that is allowed to take the value $+\infty$, and $\widehat{\pi}$ is a smooth perturbation (not necessarily concave). In such a case, $\widehat{\beta}$ is supposed to be proper and lower semicontinuous so that its subdifferential is well defined while the derivative might

not exist. This happens, for example, in the case (1.6) and then equation (1.2) becomes a differential inclusion. Finally, the operator $\text{sign} : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is defined by

$$\text{sign } r := \frac{r}{|r|} \quad \text{if } r \neq 0 \quad \text{and} \quad \text{sign } 0 := [-1, 1]. \quad (1.7)$$

The aim of introducing such a feedback law in (1.2) is to force the order parameter to reach a prescribed distribution φ^* in a finite time. However, the resulting problem of forcing the solution of the modified system to reach the manifold $\varphi = \varphi^*$ in a finite time looks difficult. Indeed, we can ensure the existence of the desired sliding mode only under a suitable compatibility condition between the measure of the set Ω and the viscosity parameter τ (cf. the following (1.9)).

The above system is complemented by initial conditions like $\varphi(0) = \varphi_0$ and $\sigma(0) = \sigma_0$ and suitable boundary conditions. Concerning the latter, we take the usual homogeneous Neumann conditions for φ and σ , that is,

$$\partial_n \varphi = 0 \quad \text{and} \quad \partial_n \sigma = 0 \quad \text{on } \Sigma := \Gamma \times (0, T)$$

where Γ is the boundary of Ω and ∂_n is the (say, outward) normal derivative. Instead, we consider a Dirichlet boundary condition for the chemical potential, i.e.,

$$\mu = \mu_\Gamma \quad \text{on } \Sigma \quad (1.8)$$

where μ_Γ is a given smooth function. This choice is twofold: from one side it looks reasonable from the modelling point of view, in case $\mu_\Gamma = 0$, the condition (1.8) allows for the free flow of cells across the outer boundary (cf. [45] and [4] where similar conditions are imposed on a chemical potential in a different framework). On the other hand, we also need (1.8) for the analysis. Indeed, a major technical issue here, in case we choose a usual Neumann boundary condition for μ , would be to estimate its mean value. This would be doable if $\rho = 0$ and the potential F is assumed to have a controlled growth (cf., e.g., [27]), but it is not the case when the feedback law is added in (1.2) and under our fully general assumptions on F .

At this point, without the aim of completeness, let us describe the recent literature both on tumor growth modelling and on SMC related to our problem.

Modelling tumor growth dynamics has recently become a major issue in applied mathematics (see, e.g., [15, 45]). Numerical simulations of diffuse interface models for tumor growth have been carried out in several papers (see, e.g., [15, Ch. 8]); nonetheless, a rigorous mathematical analysis of the resulting systems of PDEs is still in its infancy (cf., e.g., [9–11, 17, 21–25]). Recently, in [45] the authors introduced a continuum diffuse interface model of multispecies tumor growth and tumor-induced angiogenesis in two and three dimensions for investigating their morphological evolution. They make use of the Cahn–Hilliard framework which originated from the theory of phase transitions, and which is used extensively in materials science and multiphase fluid flow. Other diffuse interface models including chemotaxis and transport effects have been subsequently introduced (cf. [26, 28]) and also the formal sharp interface limits have been investigated. Rigorous sharp interface limits have been obtained in some particular cases in the two recent works [34, 40].

Regarding the SMC literature, the SMC scheme is well known for its robustness against variations of dynamics, disturbances, time-delays and nonlinearities. The design procedure of a SMC system is a two-stage process. The first phase is to choose a set of sliding manifolds such that the original system restricted to the intersection of them has a desired behavior. In this paper, we choose to force the tumor phase parameter to stay constant in time within finite time with the obvious application in mind that the phase φ should become as close as possible to the constant value $\varphi = -1$ corresponding to the case when no tumorous phase is present anymore or to a configuration φ^* which is suitable for surgery. The second step is to design a SMC law that forces the system trajectories to stay onto the sliding surface. To this end, we have added the term $\rho \operatorname{sign}(\varphi - \varphi^*)$ in the Cahn–Hilliard evolution for φ (cf. (1.2)) in order to force φ to stay equal to a given desired value φ^* in a finite time.

Sliding mode controls are pretty attractive in many applications. As a result, in recent years there has been a growing interest in extending well-developed methods for finite-dimensional systems described by ODEs (see, for example, [32, 35–37]) or also to control infinite-dimensional dynamical systems (cf. [37–39]). Moreover, the theoretical development in Hilbert spaces or for PDE systems has only taken the attention in the last ten years. In this regard, we can cite the papers [8, 33, 44] concerning the control of semilinear PDE systems.

Finally, we can quote the recent contribution [3], where a sliding mode approach is applied for the first time to phase field systems of Caginalp type coupling the evolution of a phase parameter to the one of the relative temperature, and the chosen SMC laws force the system to reach within finite time the sliding manifold. In that case it was possible to have different choices for the manifold: in particular, either one of the physical variables or a combination of them could remain constant in time. With reference to the results of [3], we aim to mention the analyses developed in [13, 14]: in particular, the second contribution is devoted to a conserved phase field system with a SMC feedback law for the internal energy in the temperature equation.

In the present contribution, instead, we are forced, mainly from the fact that we have a fourth order equation for φ , to include a sliding mode control of the type $\rho \operatorname{sign}(\varphi - \varphi^*)$ in the chemical potential μ (cf. (1.2)) and we cannot handle neither the presence of the σ -dependence in it or a non-local in space control law (as we did in some cases in [3]). We need here, for technical reasons, to include a local in space control, i.e., such that its value at any point and any time just depends on the value of the state. In this way, however, as already mentioned above, we need to enforce a compatibility condition between the size of $|\Omega|$ and the viscosity coefficient τ of the type

$$2C_{sh} |\Omega|^{2/3} < \tau \tag{1.9}$$

where C_{sh} is the constant related to some embedding inequalities (cf. (2.29) in the next Section 2). Such a condition means that either $|\Omega|$ has to be sufficiently small once the shape of Ω is fixed in the sense of the following Remark 2.1 or τ must be sufficiently large compared to the size of $|\Omega|$. Regarding the fact that we do not treat here a feedback law depending also on σ , this turns out to be quite reasonable in view of applications since we mainly aim to optimize the tumor cell distributions and not the nutrient concentration, in general.

Other approaches to the problem of control for tumor growth models are possible, even if a few mathematical results are presently available on this subject in the literature. In the two recent papers [27] and [12] the authors face the problem of finding first order necessary optimality conditions for the minimization of a cost functional forcing the phase to approach the desired target φ^* in the best possible way by means of a control variable representing the concentration of cytotoxic drugs in [27] and the supply of a nutrient or a drug in a chemotherapy, in [12] (it could be the function g in (1.3) in the present contribution).

The main advantage of controlling the sliding mode is that it strengthens the trajectories of the system to reach the sliding surface and keep it on it in a pointwise way, while, in general, within the classical optimal control theory (cf., e.g., [12, 27]), one can get just necessary optimality conditions and the control is nonlocal in space and/or in time.

The paper is organized as follows. In the next section, we list our assumptions, state the problem in a precise form and present our results. The last two sections are devoted to the corresponding proofs. Section 3 deals with well-posedness, while the existence of the sliding mode is proved in Section 4.

2 Statement of the problem and results

In this section, we describe the problem under study and present our results. As in the Introduction, Ω is the body where the evolution takes place. We assume $\Omega \subset \mathbb{R}^3$ to be open, bounded, connected, and smooth, and we write $|\Omega|$ for its Lebesgue measure. Moreover, Γ and ∂_n still stand for the boundary of Ω and the outward normal derivative, respectively. Given a finite final time $T > 0$, we set for convenience $Q := \Omega \times (0, T)$. Furthermore, if X is a Banach space, the symbol $\|\cdot\|_X$ denotes its norm, with the exception of the norms in the L^∞ spaces on Ω , Q and Σ , for which we use the same symbol $\|\cdot\|_\infty$ since no confusion can arise. Finally, the dual space of X is denoted by X^* and we write $\langle \cdot, \cdot \rangle_{X^*, X}$ for the duality pairing between X^* and X .

Now, we specify the assumptions on the structure of our system. We assume that

$$\gamma_i \in [0, +\infty) \quad \text{for } i = 1, 2, 3, \quad \gamma_4, \tau \in (0, +\infty) \quad \text{and} \quad \sigma_s \in \mathbb{R} \quad (2.1)$$

$$\widehat{\beta} : \mathbb{R} \rightarrow [0, +\infty] \quad \text{is convex, proper and l.s.c. with } \widehat{\beta}(0) = 0 \quad (2.2)$$

$$\widehat{\pi} : \mathbb{R} \rightarrow \mathbb{R} \quad \text{is a } C^1 \text{ function and } \widehat{\pi}' \text{ is Lipschitz continuous,} \quad (2.3)$$

$$p : \mathbb{R} \rightarrow [0, +\infty) \quad \text{is a bounded and Lipschitz continuous function.} \quad (2.4)$$

We set for brevity

$$\beta := \partial \widehat{\beta}, \quad \pi := \widehat{\pi}', \quad L_\pi = \text{the Lipschitz constant of } \pi \quad (2.5)$$

and denote by $D(\beta)$ and $D(\widehat{\beta})$ the effective domains of β and $\widehat{\beta}$, respectively. Next, in order to simplify notations, we set

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad V_0 := H_0^1(\Omega) \quad (2.6)$$

$$W := \{v \in H^2(\Omega) : \partial_n v = 0\} \quad \text{and} \quad W_0 := H^2(\Omega) \cap H_0^1(\Omega) \quad (2.7)$$

and endow these spaces with their standard norms. Moreover, we denote by C_{sh} a constant realizing the inequalities

$$\|v\|_\infty \leq C_{sh} |\Omega|^{1/6} \|\Delta v\|_H \quad \text{for every } v \in W_0 \quad (2.8)$$

$$\|v\|_\infty \leq C_{sh} (|\Omega|^{-1/2} \|v\|_H + |\Omega|^{1/6} \|\Delta v\|_H) \quad \text{for every } v \in W. \quad (2.9)$$

Remark 2.1. We show that such a constant actually exists and depends on Ω just through its shape. Hence, we consider a class of open sets having the same shape. To do this, we fix an open set $\Omega_0 \subset \mathbb{R}^3$ with Lebesgue measure 1. Then, each open set of the same class is related to Ω_0 by the formula $\Omega = \{x_0\} + \lambda R \Omega_0$, where x_0 is a point in \mathbb{R}^3 , the real number λ is positive and R belongs to the rotation group $SO(3)$. Let now C_{sh} be a constant satisfying (2.8)–(2.9) with Ω and $|\Omega|$ replaced by Ω_0 and 1, respectively. Such a constant exists since the three-dimensional open set Ω_0 is supposed to be bounded and smooth, as usual. Now, we take any $v \in W$ and check (2.9). We define v_0 belonging to the analogue of W constructed on Ω_0 , i.e., $v_0 \in H^2(\Omega_0)$ with $\partial_n v_0 = 0$, by the formula $v_0(y) := v(x_0 + \lambda R y)$ for $y \in \Omega_0$. Then, it is straightforward to show that

$$\|v_0\|_{L^\infty(\Omega_0)} = \|v\|_{L^\infty(\Omega)} \quad \text{and} \quad \|v_0\|_{L^2(\Omega_0)} = \lambda^{-3/2} \|v\|_{L^2(\Omega)} \quad (2.10)$$

$$\|\Delta v_0\|_{L^2(\Omega_0)} = \lambda^{1/2} \|\Delta v\|_{L^2(\Omega)}. \quad (2.11)$$

On the other hand, we have that $\lambda = |\Omega|^{1/3}$ (choose $v \equiv 1$, whence $v_0 \equiv 1$, in the second (2.10)). Therefore, we deduce that

$$\begin{aligned} \|v\|_{L^\infty(\Omega)} &= \|v_0\|_{L^\infty(\Omega_0)} \leq C_{sh} (\|v_0\|_{L^2(\Omega_0)} + \|\Delta v_0\|_{L^2(\Omega_0)}) \\ &= C_{sh} (|\Omega|^{-1/2} \|v\|_{L^2(\Omega)} + |\Omega|^{1/6} \|\Delta v\|_{L^2(\Omega)}) \end{aligned}$$

i.e., (2.9). The derivation of (2.8) is similar and even simpler.

At this point, we describe the state system modified by the state-feedback control law. We introduce the operator $\text{sign} : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ defined by

$$\text{sign } r := \frac{r}{|r|} \quad \text{if } r \neq 0 \quad \text{and} \quad \text{sign } 0 := [-1, 1]. \quad (2.12)$$

Notice that sign is the subdifferential of the real function $r \mapsto |r|$ and thus is maximal monotone. Next, we reduce the Dirichlet boundary condition $\mu = \mu_\Gamma$ to the homogeneous one. By assuming $\mu_\Gamma \in L^2(0, T; H^{1/2}(\Gamma))$ just to start with, we introduce the harmonic extension $\mu_{\mathcal{H}}$ of μ_Γ defined in this way: for a.a. $t \in (0, T)$, $\mu_{\mathcal{H}}(t)$ is the unique solution to the problem

$$\mu_{\mathcal{H}}(t) \in H^1(\Omega), \quad -\Delta \mu_{\mathcal{H}}(t) = 0 \quad \text{in } \mathcal{D}'(Q) \quad \text{and} \quad \mu_{\mathcal{H}}(t)|_\Gamma = \mu_\Gamma(t). \quad (2.13)$$

Then, we take $\mu - \mu_{\mathcal{H}}$ as new unknown. However, in order to avoid a new notation, we still term μ the above difference. Thus, the problem to be solved is the following: we are given the functions g , μ_Γ , φ^* and the initial data φ_0 and σ_0 such that

$$g \in L^\infty(Q), \quad \mu_\Gamma \in H^1(0, T; L^2(\Gamma)) \cap L^\infty(0, T; H^{3/2}(\Gamma)) \quad (2.14)$$

$$\varphi^* \in W \quad \text{and} \quad \inf D(\beta) < \inf \varphi^* \leq \sup \varphi^* < \sup D(\beta) \quad (2.15)$$

$$\varphi_0 \in W, \quad \beta^\circ(\varphi_0) \in H \quad \text{and} \quad \sigma_0 \in V \cap L^\infty(\Omega) \quad (2.16)$$

where β° denotes the minimal section of β . Notice that the assumptions on μ_Γ in (2.14) are additional and do not follow from what is previously stated on $\mu_{\mathcal{H}}$. Then, we look for a quintuplet $(\varphi, \mu, \sigma, \xi, \zeta)$ satisfying the regularity requirements

$$\varphi \in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \quad (2.17)$$

$$\mu \in L^\infty(0, T; W_0) \quad (2.18)$$

$$\sigma \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (2.19)$$

$$\xi \in L^\infty(0, T; H) \quad \text{and} \quad \zeta \in L^\infty(0, T; H) \quad (2.20)$$

and solving

$$\partial_t \varphi - \Delta \mu = (\gamma_1 \sigma - \gamma_2) p(\varphi) \quad \text{a.e. in } Q \quad (2.21)$$

$$\mu = \tau \partial_t \varphi - \Delta \varphi + \xi + \pi(\varphi) + \rho \zeta - \mu_{\mathcal{H}} \quad \text{a.e. in } Q \quad (2.22)$$

$$\partial_t \sigma - \Delta \sigma = -\gamma_3 \sigma p(\varphi) + \gamma_4 (\sigma_s - \sigma) + g \quad \text{a.e. in } Q \quad (2.23)$$

$$\xi \in \beta(\varphi) \quad \text{and} \quad \zeta \in \text{sign}(\varphi - \varphi^*) \quad \text{a.e. in } Q \quad (2.24)$$

$$\varphi(0) = \varphi_0 \quad \text{and} \quad \sigma(0) = \sigma_0 \quad (2.25)$$

where ρ is a positive parameter. We notice that the boundary conditions $\partial_n \varphi = 0$, $\mu = 0$ and $\partial_n \sigma = 0$ are contained in (2.17), (2.18) and (2.19), respectively, due to the definitions (2.7) of W and W_0 . We also remark that

$$\mu_\Gamma \in L^\infty(\Sigma) \quad (2.26)$$

as a consequence of (2.14). Indeed, $H^{3/2}(\Gamma) \subset L^\infty(\Gamma)$ since Σ is a two-dimensional smooth surface. Here is our well-posedness result.

Theorem 2.2. *Assume (2.1)–(2.4) and (2.14)–(2.16). Then, for every $\rho > 0$, there exists at least one quintuplet $(\varphi, \mu, \sigma, \xi, \zeta)$ fulfilling (2.17)–(2.20), solving problem (2.21)–(2.25) and satisfying the estimates*

$$\|\mu\|_\infty \leq C_{sh} \frac{2|\Omega|^{2/3}}{\tau} \rho + \widehat{C} \quad (2.27)$$

$$|\sigma| \leq \sigma^* := \max\{\|\sigma_s + \gamma_4^{-1}g\|_\infty, \|\sigma_0\|_\infty\} \quad \text{a.e. in } Q \quad (2.28)$$

where C_{sh} is the same as in (2.8)–(2.9) and the constant \widehat{C} depends only on Ω , T and the quantities involved in assumptions (2.1)–(2.4) and (2.14)–(2.16). In particular, \widehat{C} does not depend on ρ . Moreover, the components φ and σ of the solution are uniquely determined.

The above result is quite general. In particular, all the potentials (1.4)–(1.6) are certainly allowed.

As far as the problem of a sliding mode is concerned, we prove a result that only involves the component φ of the solution, which is uniquely determined. However, we can ensure the existence of a sliding mode at least for ρ large enough only under a restriction. Namely, we need the following condition

$$C_{sys} := C_{sh} \frac{2|\Omega|^{2/3}}{\tau} < 1 \quad (2.29)$$

where C_{sh} is the constant that appears in (2.8)–(2.9). Such a condition means that $|\Omega|$ has to be sufficiently small once the shape of Ω is fixed in the sense of Remark 2.1.

Theorem 2.3. *In addition to (2.1)–(2.4) and (2.14)–(2.16), assume that*

$$\Delta\varphi^* \in L^\infty(\Omega). \quad (2.30)$$

Moreover, assume (2.29). Then, there exists $\rho^* > 0$, depending only on Ω , T , the structure and the data of the problem, such that, for every $\rho > \rho^*$, the following is true: if $(\varphi, \mu, \sigma, \xi, \zeta)$ is a solution to problem (2.21)–(2.25) whose component μ satisfies (2.27), there exists a time $T^* \in [0, T)$ such that

$$\varphi(t) = \varphi^* \quad \text{a.e. in } \Omega \quad \text{for every } t \in [T^*, T]. \quad (2.31)$$

In particular, there exists a solution for which (2.31) holds true.

Remark 2.4. In the proof we give in Section 4, we show that possible values of ρ^* and T^* that fit the above statement are

$$\rho^* = \frac{1}{1 - C_{sys}} \left(\widehat{C} + M + M_\pi^* + \frac{\tau}{T} M_0 \right) \quad \text{and} \quad T^* = \frac{\tau}{\rho - A(\rho)} M_0$$

where M , M_0 , M_π^* and $A(\rho)$ are given by

$$\begin{aligned} M &:= \|\mu_\Gamma\|_\infty + \|\Delta\varphi^*\|_\infty + \|\xi^*\|_\infty, & M_0 &:= \|\varphi_0 - \varphi^*\|_\infty \\ M_\pi^* &:= \sup\{|\pi(\varphi^*(x) + r)| : x \in \Omega, |r| \leq M_0\} \\ A(\rho) &:= C_{sys} \rho + \widehat{C} + M + M_\pi^* \quad \text{for } \rho > 0 \end{aligned}$$

where $\xi^* := \beta^\circ(\varphi^*)$. In these formulas, \widehat{C} is the same as in (2.27). We will see that the above definitions ensure that $A(\rho) < \rho$ for $\rho > \rho^*$ and that $T^* \in [0, T)$.

The rest of the section is devoted to make some notations precise and to introduce some tools we use in the remainder of the paper. In performing our a priori estimates, we often account for the Hölder inequality and the Young inequality

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \geq 0 \text{ and } \delta > 0. \quad (2.32)$$

Moreover, we repeatedly use the notation

$$Q_t := \Omega \times (0, t) \quad \text{and} \quad \Sigma_t := \Gamma \times (0, t) \quad \text{for } t \in (0, T). \quad (2.33)$$

For simplicity, we usually omit dx , ds , etc. in integrals. More precisely, we explicitly write, e.g., ds only if the variable s actually appears in the function under the integral sign. We also take advantage of the Dirichlet problem solver operator $\mathcal{D} : H^{-1}(\Omega) \rightarrow H_0^1(\Omega)$ defined as follows: if $f \in H^{-1}(\Omega)$, then $\mathcal{D}f$ is the unique solution u to the Dirichlet problem

$$u \in H_0^1(\Omega) \quad \text{and} \quad -\Delta u = f. \quad (2.34)$$

As Ω is bounded and smooth, we have that $\mathcal{D}f \in W_0$ as well as

$$\|\mathcal{D}f\|_{H^2(\Omega)} \leq C\|f\|_H, \quad \text{in particular,} \quad \|\mathcal{D}f\|_H \leq C\|f\|_H \quad \text{for every } f \in H \quad (2.35)$$

where C depends only on Ω . Furthermore, we define the equivalent norm in $H^{-1}(\Omega)$ by the formula

$$\|f\|_*^2 := \int_{\Omega} |\nabla \mathcal{D}f|^2 \quad \text{for } f \in H^{-1}(\Omega). \quad (2.36)$$

Notice that

$$\langle f, \mathcal{D}f \rangle_{V_0^*, V_0} = \|f\|_*^2 \quad \text{and} \quad \langle f(t), \mathcal{D}(\partial_t f(t)) \rangle_{V_0^*, V_0} = \frac{1}{2} \frac{d}{dt} \|f(t)\|_*^2, \quad t \in (0, T), \quad (2.37)$$

for every $f \in H^{-1}(\Omega)$ and $f \in H^1(0, T; H^{-1}(\Omega))$, respectively. For the same reason as above, the assumptions on μ_{Γ} in (2.14) imply a proper regularity for $\mu_{\mathcal{H}}$ and the corresponding estimates. Namely, owing to the maximum principle and to the elliptic regularity results stated, e.g., in [6, Thm. 3.2, p. 1.79], we have that

$$\|\mu_{\mathcal{H}}\|_{\infty} \leq \|\mu_{\Gamma}\|_{\infty} \quad (2.38)$$

$$\|\mu_{\mathcal{H}}\|_{L^{\infty}(0, T; H^2(\Omega))} \leq C \|\mu_{\Gamma}\|_{L^{\infty}(0, T; H^{3/2}(\Gamma))} \quad (2.39)$$

$$\|\partial_t \mu_{\mathcal{H}}\|_{L^2(0, T; H)} \leq C \|\partial_t \mu_{\Gamma}\|_{L^2(0, T; L^2(\Gamma))} \quad (2.40)$$

with a constant C that depends only on Ω .

Finally, while a particular care is taken in computing some constants, we follow a general rule to denote less important ones, in order to avoid boring calculations. The small-case symbol c stands for different constants which depend only on Ω , on the final time T , the shape of the nonlinearities and on the constants and the norms of the functions involved in the assumptions of our statements, but ρ . The dependence on ρ will be always written explicitly, indeed. Constants depending on further parameters are characterized by a corresponding subscript. So, e.g., we write c_{ε} for constants depending on ε as well. Hence, the meaning of c and c_{ε} might change from line to line and even in the same chain of equalities or inequalities. On the contrary, we mark precise constants which we can refer to by using different symbols, e.g., capital letters, mainly with indices.

3 Well-posedness

This section is devoted to the proof of Theorem 2.2. We first prove the partial uniqueness given in the statement. Then, we introduce a proper regularization and construct a solution satisfying both (2.27) and the maximum principle (2.28). It is convenient to observe once and for all that every solution satisfies

$$\mathcal{D}(\partial_t \varphi) + \tau \partial_t \varphi - \Delta \varphi + \xi + \pi(\varphi) + \rho \zeta - \mu_{\mathcal{H}} = \mathcal{D}((\gamma_1 \sigma - \gamma_2) p(\varphi)) \quad \text{a.e. in } Q. \quad (3.1)$$

Indeed, it suffices to apply \mathcal{D} to both sides of (2.21) (by recalling that μ is V_0 -valued) and replace μ by means of (2.22).

Partial uniqueness. We pick two solutions $(\varphi_i, \mu_i, \sigma_i, \xi_i, \zeta_i)$, $i = 1, 2$, corresponding to the same data and prove that $(\varphi_1, \sigma_1) = (\varphi_2, \sigma_2)$. We write (3.1) and (2.23) for both solutions and take the difference. By setting for brevity $\varphi := \varphi_1 - \varphi_2$ and defining the analogous differences μ, σ, ξ and ζ , we have that a.e. in Q

$$\begin{aligned} & \mathcal{D}(\partial_t \varphi) + \tau \partial_t \varphi - \Delta \varphi + \xi + \rho \zeta \\ &= \pi(\varphi_2) - \pi(\varphi_1) + \mathcal{D}(\gamma_1 \sigma p(\varphi_1)) + \mathcal{D}((\gamma_1 \sigma_2 - \gamma_2)(p(\varphi_1) - p(\varphi_2))) \\ & \partial_t \sigma - \Delta \sigma + \gamma_4 \sigma = -\gamma_3 \sigma p(\varphi_1) - \gamma_3 \sigma_2 (p(\varphi_1) - p(\varphi_2)). \end{aligned}$$

Now, we multiply such equations by φ and σ , respectively, sum up and integrate over Q_t , where $t \in (0, T)$ is arbitrary. Thanks to the boundary and initial conditions, the definition (2.36) of $\|\cdot\|_*$, the second inequality in (2.35), the Lipschitz continuity of π and p , the boundedness of p , and the inequality (2.28) satisfied by σ_1 and σ_2 , we obtain

$$\begin{aligned} & \frac{1}{2} \|\varphi(t)\|_*^2 + \frac{\tau}{2} \|\varphi(t)\|_H^2 + \int_{Q_t} |\nabla \varphi|^2 + \int_{Q_t} \xi \varphi + \rho \int_{Q_t} \zeta \varphi \\ & \quad + \frac{1}{2} \|\sigma(t)\|_H^2 + \int_{Q_t} |\nabla \sigma|^2 + \gamma_4 \int_{Q_t} |\sigma|^2 \\ & \leq c \int_{Q_t} (|\varphi|^2 + |\varphi| |\sigma| + |\sigma|^2) \leq c \int_{Q_t} (|\varphi|^2 + |\sigma|^2) \end{aligned}$$

where the values of c might depend on the solutions we are considering as well. All the terms on the left-hand side are nonnegative (those involving ξ and ζ by monotonicity). Therefore, by applying the Gronwall lemma, we conclude that $\varphi = \sigma = 0$, i.e., that $(\varphi_1, \sigma_1) = (\varphi_2, \sigma_2)$.

At this point, we start proving the existence of a solution satisfying the properties of the statements. To this end, we introduce the approximating problem obtained by replacing the graphs sign and β by the Lipschitz continuous functions sign_ε and β_ε , their Yosida regularizations at the level $\varepsilon \in (0, 1)$ (concerning general properties of maximal monotone and subdifferential operators along with their Yosida regularizations, the reader can see, e.g., [2, 5]). We also introduce their primitives $|\cdot|_\varepsilon$ and $\widehat{\beta}_\varepsilon$ vanishing at the origin. Such functions satisfy the properties listed below (the last one also being a consequence of the assumptions (2.2)):

$$\text{sign}_\varepsilon r = \frac{r}{\max\{\varepsilon, |r|\}} \quad \text{for every } r \in \mathbb{R} \quad (3.2)$$

$$0 \leq |r|_\varepsilon := \int_0^r \text{sign}_\varepsilon s \, ds \leq |r| \quad \text{for every } r \in \mathbb{R} \quad (3.3)$$

$$|\beta_\varepsilon(r)| \leq |\beta^\circ(r)| \quad \text{for every } r \in D(\beta), \quad \text{where} \quad (3.4)$$

$\beta^\circ(r)$ is the element of $\beta(r)$ having minimum modulus

$$0 \leq \widehat{\beta}_\varepsilon(r) := \int_0^r \beta_\varepsilon(s) \, ds \leq \widehat{\beta}(r) \quad \text{for every } r \in D(\widehat{\beta}). \quad (3.5)$$

Moreover, we choose the following regularization $I_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ of the identity:

$$I_\varepsilon(r) := \max\{-1/\varepsilon, \min\{r, 1/\varepsilon\}\} \quad \text{for } r \in \mathbb{R}. \quad (3.6)$$

Thus, the approximating problem consists in finding a triplet $(\varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)$ satisfying (2.17)–(2.20) and solving

$$\partial_t \varphi_\varepsilon - \Delta \mu_\varepsilon = (\gamma_1 I_\varepsilon(\sigma_\varepsilon) - \gamma_2) p(\varphi_\varepsilon) \quad \text{a.e. in } Q \quad (3.7)$$

$$\mu_\varepsilon = \tau \partial_t \varphi_\varepsilon - \Delta \varphi_\varepsilon + \xi_\varepsilon + \pi(\varphi_\varepsilon) + \rho \zeta_\varepsilon - \mu_{\mathcal{H}} \quad \text{a.e. in } Q \quad (3.8)$$

$$\partial_t \sigma_\varepsilon - \Delta \sigma_\varepsilon = -\gamma_3 I_\varepsilon(\sigma_\varepsilon) p(\varphi_\varepsilon) + \gamma_4 (\sigma_s - \sigma_\varepsilon) + g \quad \text{a.e. in } Q \quad (3.9)$$

$$\text{where } \xi_\varepsilon := \beta_\varepsilon(\varphi_\varepsilon) \quad \text{and} \quad \zeta_\varepsilon := \text{sign}_\varepsilon(\varphi_\varepsilon - \varphi^*) \quad (3.10)$$

$$\varphi_\varepsilon(0) = \varphi_0 \quad \text{and} \quad \sigma_\varepsilon(0) = \sigma_0. \quad (3.11)$$

Also in this case, the boundary conditions are contained in the regularity requirements.

Theorem 3.1. *Under the assumptions of Theorem 2.2, the problem (3.7)–(3.11) has a unique solution $(\varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)$ satisfying the regularity requirements (2.17)–(2.20). Moreover, the component σ_ε fulfils the inequality*

$$|\sigma_\varepsilon| \leq \sigma^* \quad \text{a.e. in } Q \quad (3.12)$$

with the same σ^* as in (2.28).

Proof. We show the well-posedness of the approximating problem in a less regular framework. Namely, we look for solutions satisfying

$$\varphi_\varepsilon \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (3.13)$$

$$\mu_\varepsilon \in L^2(0, T; W_0) \quad (3.14)$$

$$\sigma_\varepsilon \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (3.15)$$

instead of (2.17)–(2.20). In the sequel of the paper, it will be clear that the solution we find here also satisfies the regularity we have required, even though we will proceed formally. We present the approximating problem in the equivalent form obtained by replacing (3.7) by the analogue of (3.1), i.e.,

$$\mathcal{D}(\partial_t \varphi_\varepsilon) + \tau \partial_t \varphi_\varepsilon - \Delta \varphi_\varepsilon + \xi_\varepsilon + \pi(\varphi_\varepsilon) + \rho \zeta_\varepsilon - \mu_{\mathcal{H}} = \mathcal{D}((\gamma_1 I_\varepsilon(\sigma_\varepsilon) - \gamma_2) p(\varphi_\varepsilon)) \quad (3.16)$$

where ξ_ε and ζ_ε are still given by (3.10). We want to show that we can solve (3.16), (3.9) and (3.11) for $(\varphi_\varepsilon, \sigma_\varepsilon)$ with the proper regularity that is needed. Then, we will use (3.8) as a definition of μ_ε . In order to solve the sub-problem, we present it as a Cauchy problem for a nonlinear abstract equation of the type

$$\frac{d}{dt}(\varphi_\varepsilon, \sigma_\varepsilon) + \mathbb{A}(\varphi_\varepsilon, \sigma_\varepsilon) + \mathbb{F}(\varphi_\varepsilon, \sigma_\varepsilon) = G \quad (3.17)$$

in the framework of the Hilbert triplet $(\mathbb{V}, \mathbb{H}, \mathbb{V}^*)$ where

$$\mathbb{V} := V \times V \quad \text{and} \quad \mathbb{H} := H \times H \quad (3.18)$$

with a non-standard embedding $\mathbb{H} \subset \mathbb{V}^*$. To justify what we are going to write, in particular the forthcoming definitions (3.19) and (3.21)–(3.22), we first notice that \mathcal{D} is a symmetric linear continuous operator from H into itself which satisfies the first property in (2.37). This implies that the integral $\int_\Omega (\mathcal{D}v)v$ is nonnegative for every $v \in H$, so that the operator $\mathcal{D} + \tau I$, where I is the identity map of H , is an isomorphism from H onto itself and $(\mathcal{D} + \tau I)^{-1}$ is a well-defined linear continuous operator from H into itself. Once this is established, we observe that the system given by (3.16) and (3.9)–(3.10) is equivalent to the following variational equation

$$\begin{aligned} & \int_\Omega (\mathcal{D}(\partial_t \varphi_\varepsilon) + \tau \partial_t \varphi_\varepsilon)v + \int_\Omega \partial_t \sigma_\varepsilon z + \int_\Omega \nabla \varphi_\varepsilon \cdot \nabla v + \int_\Omega \nabla \sigma_\varepsilon \cdot \nabla z \\ & + \int_\Omega (\mathcal{D} + \tau I)(\mathcal{D} + \tau I)^{-1} \left((\beta_\varepsilon + \pi + \rho \operatorname{sign}_\varepsilon)(\varphi_\varepsilon) - \mathcal{D}((\gamma_1 I_\varepsilon(\sigma_\varepsilon) - \gamma_2) p(\varphi_\varepsilon)) \right) v \\ & + \int_\Omega (\gamma_3 I_\varepsilon(\sigma_\varepsilon) p(\varphi_\varepsilon) - \gamma_4(\sigma_s - \sigma_\varepsilon))z \\ & = \int_\Omega ((\mathcal{D} + \tau I)(\mathcal{D} + \tau I)^{-1} \mu_{\mathcal{H}}) v + \int_\Omega g z \end{aligned}$$

holding a.e. in $(0, T)$, for every $(v, z) \in V \times V$. This equation can be written as

$$\begin{aligned} & (\partial_t(\varphi_\varepsilon, \sigma_\varepsilon), (v, z))_{\mathbb{H}} + \langle \mathbb{A}(\varphi_\varepsilon, \sigma_\varepsilon), (v, z) \rangle_{\mathbb{V}^*, \mathbb{V}} + (\mathbb{F}(\varphi_\varepsilon, \sigma_\varepsilon), (v, z))_{\mathbb{H}} \\ & = (((\mathcal{D} + \tau I)^{-1} \mu_{\mathcal{H}}, g), (v, z))_{\mathbb{H}} \end{aligned}$$

that is, in the form (3.17) with an obvious meaning of G , provided that $(\cdot, \cdot)_{\mathbb{H}}$ and $\mathbb{A} : \mathbb{V} \rightarrow \mathbb{V}^*$ are given by

$$((\widehat{\varphi}, \widehat{\sigma}), (v, z))_{\mathbb{H}} := \int_{\Omega} (\mathcal{D}\widehat{\varphi} + \tau\widehat{\varphi})v + \int_{\Omega} \widehat{\sigma}z \quad \text{for } (\widehat{\varphi}, \widehat{\sigma}), (v, z) \in \mathbb{H} \quad (3.19)$$

$$\langle \mathbb{A}(\widehat{\varphi}, \widehat{\sigma}), (v, z) \rangle_{\mathbb{V}^*, \mathbb{V}} := \int_{\Omega} \nabla \widehat{\varphi} \cdot \nabla v + \int_{\Omega} \nabla \widehat{\sigma} \cdot \nabla z \quad \text{for } (\widehat{\varphi}, \widehat{\sigma}), (v, z) \in \mathbb{V} \quad (3.20)$$

and the function $\mathbb{F} : \mathbb{H} \rightarrow \mathbb{H}$ is defined by the following rule: for $(\widehat{\varphi}, \widehat{\sigma}) \in \mathbb{H}$, the value $\mathbb{F}(\widehat{\varphi}, \widehat{\sigma})$ is the pair $(\mathbb{F}_1(\widehat{\varphi}, \widehat{\sigma}), \mathbb{F}_2(\widehat{\varphi}, \widehat{\sigma}))$ given by

$$\mathbb{F}_1(\widehat{\varphi}, \widehat{\sigma}) := (\mathcal{D} + \tau I)^{-1}((\beta_\varepsilon + \pi + \rho \operatorname{sign}_\varepsilon)(\widehat{\varphi}) - \mathcal{D}((\gamma_1 I_\varepsilon(\widehat{\sigma}) - \gamma_2)p(\widehat{\varphi}))) \quad (3.21)$$

$$\mathbb{F}_2(\widehat{\varphi}, \widehat{\sigma}) := \gamma_3 I_\varepsilon(\widehat{\sigma})p(\widehat{\varphi}) - \gamma_4(\sigma_s - \widehat{\sigma}). \quad (3.22)$$

By owing to our preliminary observations, we see that, from one side, formula (3.19) actually yields an inner product in \mathbb{H} that is equivalent to the standard one; on the other side, by also noting that the real functions

$$r \mapsto (\beta_\varepsilon + \pi + \rho \operatorname{sign}_\varepsilon)(r), \quad r \in \mathbb{R}, \quad \text{and} \quad (r, s) \mapsto I_\varepsilon(s)p(r), \quad (r, s) \in \mathbb{R}^2$$

are Lipschitz continuous (cf. (2.4)), we conclude that $\mathbb{F} : \mathbb{H} \rightarrow \mathbb{H}$ is Lipschitz continuous. Note that, in particular, the boundedness of p is used in order to check the Lipschitz continuity of the considered function of (r, s) . In addition, we observe at once that the meaning of the embedding $\mathbb{I} : \mathbb{H} \rightarrow \mathbb{V}^*$ is the following: for arbitrary $(\widehat{\varphi}, \widehat{\sigma}) \in \mathbb{H}$ and $(v, z) \in \mathbb{V}$, we have that

$$\langle (\widehat{\varphi}, \widehat{\sigma}), (v, z) \rangle_{\mathbb{V}^*, \mathbb{V}} = ((\widehat{\varphi}, \widehat{\sigma}), (v, z))_{\mathbb{H}} = \int_{\Omega} (\mathcal{D}\widehat{\varphi} + \tau\widehat{\varphi})v + \int_{\Omega} \widehat{\sigma}z.$$

Now, we show that $\mathbb{A} : \mathbb{V} \rightarrow \mathbb{V}^*$ is maximal monotone. Indeed, it is monotone since

$$\langle \mathbb{A}(v, z), (v, z) \rangle_{\mathbb{V}^*, \mathbb{V}} = \int_{\Omega} (|\nabla v|^2 + |\nabla z|^2) \geq 0 \quad \text{for every } (v, z) \in \mathbb{V}.$$

On the other hand, $\mathbb{A} + \mathbb{I} : \mathbb{V} \rightarrow \mathbb{V}^*$ is surjective since it holds for every $(v, z) \in \mathbb{V}$

$$\begin{aligned} & \langle (\mathbb{A} + \mathbb{I})(v, z), (v, z) \rangle_{\mathbb{V}^*, \mathbb{V}} = \langle \mathbb{A}(v, z), (v, z) \rangle_{\mathbb{V}^*, \mathbb{V}} + ((v, z), (v, z))_{\mathbb{H}} \\ & = \int_{\Omega} (|\nabla v|^2 + |\nabla z|^2) + \int_{\Omega} (\mathcal{D}v + \tau v)v + \int_{\Omega} |z|^2 \geq \min\{1, \tau\} \|(v, z)\|_{\mathbb{V}}^2. \end{aligned}$$

We deduce that also the realization of \mathbb{A} in \mathbb{H} (or restriction of \mathbb{A} to \mathbb{H} , which we still term \mathbb{A}) defined by setting $D(\mathbb{A}) := W \times W$ is maximal monotone. The monotonicity property obviously follows. To see the maximal monotonicity, we show that, for any $G = (G_1, G_2) \in \mathbb{H}$, we can find $(\widehat{\varphi}, \widehat{\sigma}) \in W \times W$ satisfying $(\mathbb{A} + \mathbb{I})(\widehat{\varphi}, \widehat{\sigma}) = G$. Indeed,

the equation to be solved has a solution $(\widehat{\varphi}, \widehat{\sigma}) \in \mathbb{V}$. On the other hand, such a solution satisfies

$$\langle \mathbb{A}(\widehat{\varphi}, \widehat{\sigma}), (v, z) \rangle_{\mathbb{V}^*, \mathbb{V}} + ((\widehat{\varphi}, \widehat{\sigma}), (v, z))_{\mathbb{H}} = ((G_1, G_2), (v, z))_{\mathbb{H}} \quad \text{for every } (v, z) \in \mathbb{V}$$

that is the couple of variational Neumann problems

$$\begin{aligned} \int_{\Omega} \nabla \widehat{\varphi} \cdot \nabla v + \tau \int_{\Omega} \widehat{\varphi} v &= \int_{\Omega} (\mathcal{D}G_1 + \tau G_1 - \mathcal{D}\widehat{\varphi})v \quad \text{for every } v \in V \\ \int_{\Omega} \nabla \widehat{\sigma} \cdot \nabla z + \int_{\Omega} \widehat{\sigma} z &= \int_{\Omega} G_2 z \quad \text{for every } z \in V \end{aligned}$$

so that both $\widehat{\varphi}$ and $\widehat{\sigma}$ belong to W by the elliptic regularity theory. Once such a maximal monotonicity is established, we recall that the Cauchy problem to be solved is the equation (3.17) complemented by the initial condition (3.11). Thus, we can apply the general theory and ensure that such a problem has a unique solution satisfying (3.13) and (3.15). At this point, (3.8) yields μ_ε with a level of regularity that is lower than required. However, from (3.16) and (3.8) we deduce that μ_ε solves (3.7) as well, so that also the regularity requirement (3.14) holds for μ_ε . This concludes the proof of well-posedness.

Finally, we prove (3.12). In order to show the upper inequality $\sigma \leq \sigma^*$, we rewrite (3.9) in the form

$$\begin{aligned} \partial_t \sigma_\varepsilon - \Delta \sigma_\varepsilon + \gamma_3 p(\varphi_\varepsilon)(I_\varepsilon(\sigma_\varepsilon) - I_\varepsilon(\sigma^*)) + \gamma_4(\sigma_\varepsilon - \sigma^*) \\ = -\gamma_3 p(\varphi_\varepsilon) I_\varepsilon(\sigma^*) + (\gamma_4 \sigma_s + g - \gamma_4 \sigma^*) \end{aligned}$$

and observe that the right-hand side is nonpositive due to the definition of σ^* contained in (2.28) and the positivity assumptions on the constants and on p . Thus, multiplying by the positive part $v_\varepsilon := (\sigma_\varepsilon - \sigma^*)^+$, integrating over Q_t , accounting for the boundary and initial conditions, and observing that $v_\varepsilon(0) = 0$, we obtain for every $t \in [0, T]$

$$\frac{1}{2} \int_{\Omega} |v_\varepsilon(t)|^2 + \int_{Q_t} |\nabla v_\varepsilon|^2 + \gamma_3 \int_{Q_t} p(\varphi_\varepsilon)(I_\varepsilon(\sigma_\varepsilon) - I_\varepsilon(\sigma^*)) (\sigma_\varepsilon - \sigma^*)^+ + \gamma_4 \int_{Q_t} |v_\varepsilon|^2 \leq 0.$$

By also recalling the definition (3.6) of I_ε , we see that every term on the left-hand side is nonnegative. Hence, we deduce that $v_\varepsilon = 0$, i.e., $\sigma_\varepsilon \leq \sigma^*$. In order to derive the inequality $\sigma_\varepsilon \geq -\sigma^*$, one writes (2.23) as

$$\begin{aligned} \partial_t \sigma_\varepsilon - \Delta \sigma_\varepsilon + \gamma_3 p(\varphi_\varepsilon)(I_\varepsilon(\sigma_\varepsilon) + I_\varepsilon(\sigma^*)) + \gamma_4(\sigma_\varepsilon + \sigma^*) \\ = \gamma_3 p(\varphi_\varepsilon) I_\varepsilon(\sigma^*) + (\gamma_4 \sigma_s + g + \gamma_4 \sigma^*) \end{aligned}$$

and multiplies by $-(\sigma_\varepsilon + \sigma^*)^-$. □

Our project is now the following: we show that the approximating solution satisfies some bounds (in particular, we formally derive the further regularity (2.17)–(2.20) for $(\varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)$); then, we take the limit as $\varepsilon \searrow 0$ by accounting for compactness and monotonicity arguments. As far as the notation for the constants is concerned, we follow the general rule explained at the end of the previous section. In particular, the (possibly different) values denoted by c are independent of ρ and ε . The same holds for the precise constants C_1, C_2 , etc., that we introduce in the sequel and mark with capital letters for

possible future references. Moreover, due to (3.12) and the definition (3.6), from now on in (3.7), (3.9), (3.16) we can replace $I_\varepsilon(\sigma_\varepsilon)$ by σ_ε , assuming that $1/\varepsilon \geq \sigma^*$, that is, $\varepsilon \leq 1/\sigma^*$. Finally, let us note that from now on we will underline the dependence in the estimates from τ and $|\Omega|$ only when it is necessary in order to conclude the proofs.

First a priori estimate.

In view of (3.12), we notice that the right-hand side of (3.9) is uniformly bounded in $L^2(Q)$ by a constant depending only on our structural assumptions and on the norms $\|g\|_\infty$ and $\|\sigma_0\|_\infty$. By the parabolic regularity theory, we infer that

$$\|\sigma_\varepsilon\|_{H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)} \leq C_1. \quad (3.23)$$

Second a priori estimate. We add φ_ε to both sides of (3.16), then we test by $\partial_t \varphi_\varepsilon$ and integrate over $(0, t)$ by accounting for the boundary and initial conditions, and rearrange. By recalling the first property of the norm $\|\cdot\|_*$ in (2.37) and the formulas (3.5) and (3.2), we obtain

$$\begin{aligned} & \int_0^t \|\partial_t \varphi_\varepsilon(s)\|_*^2 + \tau \int_{Q_t} |\partial_t \varphi_\varepsilon|^2 + \frac{1}{2} \|\varphi_\varepsilon(t)\|_V^2 \\ & + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(t)) + \rho \int_\Omega |\varphi_\varepsilon(t) - \varphi^*|_\varepsilon \\ & = \frac{1}{2} \|\varphi_0\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_0) + \rho \int_\Omega |\varphi_0 - \varphi^*|_\varepsilon \\ & + \int_{Q_t} \mathcal{D}((\gamma_1 \sigma_\varepsilon - \gamma_2) p(\varphi_\varepsilon)) \partial_t \varphi_\varepsilon + \int_{Q_t} \mu_{\mathcal{H}} \partial_t \varphi_\varepsilon + \int_{Q_t} (\varphi_\varepsilon - \pi(\varphi_\varepsilon)) \partial_t \varphi_\varepsilon. \end{aligned}$$

Every term on the left-hand side is nonnegative. Using the Young inequality, the structural assumptions (2.14)–(2.16), (2.39), and the estimates (2.35) and (3.12), we see that the right-hand side can be bounded from above by

$$\begin{aligned} & \frac{1}{2} \|\varphi_0\|_V^2 + \int_\Omega \widehat{\beta}(\varphi_0) + \rho \int_\Omega |\varphi_0 - \varphi^*| + \frac{\tau}{2} \int_{Q_t} |\partial_t \varphi_\varepsilon|^2 \\ & + c \|\mathcal{D}((\gamma_1 \sigma_\varepsilon - \gamma_2) p(\varphi_\varepsilon))\| + \mu_{\mathcal{H}} \|L^2(Q)\| + c \int_{Q_t} (1 + |\varphi_\varepsilon|^2) \\ & \leq \frac{\tau}{2} \int_{Q_t} |\partial_t \varphi_\varepsilon|^2 + c + c \int_{Q_t} |\varphi_\varepsilon|^2. \end{aligned}$$

Hence, by applying the Gronwall lemma, we conclude that

$$\|\varphi_\varepsilon\|_{H^1(0,T;H) \cap L^\infty(0,T;V)} \leq C_2(\rho^{1/2} + 1). \quad (3.24)$$

We also get the same bound for $\widehat{\beta}_\varepsilon(\varphi_\varepsilon)$ in $L^\infty(0, T; L^1(\Omega))$.

Third a priori estimate. We (formally) differentiate (3.16) with respect to time (by accounting for (3.10)). We obtain

$$\begin{aligned} & \mathcal{D}(\partial_t^2 \varphi_\varepsilon) + \tau \partial_t^2 \varphi_\varepsilon - \Delta \partial_t \varphi_\varepsilon \\ & + \beta'_\varepsilon(\varphi_\varepsilon) \partial_t \varphi_\varepsilon + \pi'(\varphi_\varepsilon) \partial_t \varphi_\varepsilon + \rho \operatorname{sign}'_\varepsilon(\varphi_\varepsilon - \varphi^*) \partial_t(\varphi_\varepsilon - \varphi^*) - \partial_t \mu_{\mathcal{H}} \\ & = \mathcal{D}(\gamma_1 \partial_t \sigma_\varepsilon p(\varphi_\varepsilon) + (\gamma_1 \sigma_\varepsilon - \gamma_2) p'(\varphi_\varepsilon) \partial_t \varphi_\varepsilon). \end{aligned}$$

Now, we test this equation by $\partial_t \varphi_\varepsilon$ and integrate over $(0, t)$. By recalling the second identity in (2.37) and rearranging, we have that

$$\begin{aligned} & \frac{1}{2} \|\partial_t \varphi_\varepsilon(t)\|_*^2 + \frac{\tau}{2} \int_\Omega |\partial_t \varphi_\varepsilon(t)|^2 + \int_{Q_t} |\nabla \partial_t \varphi_\varepsilon|^2 \\ & \quad + \int_{Q_t} \beta'_\varepsilon(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2 + \rho \int_{Q_t} \text{sign}'_\varepsilon(\varphi_\varepsilon - \varphi^*) |\partial_t(\varphi_\varepsilon - \varphi^*)|^2 \\ & = \frac{1}{2} \|\partial_t \varphi_\varepsilon(0)\|_*^2 + \frac{\tau}{2} \int_\Omega |\partial_t \varphi_\varepsilon(0)|^2 \\ & \quad + \int_{Q_t} \left\{ \mathcal{D}(\gamma_1 \partial_t \sigma_\varepsilon p(\varphi_\varepsilon) + (\gamma_1 \sigma_\varepsilon - \gamma_2) p'(\varphi_\varepsilon) \partial_t \varphi_\varepsilon) \right\} \partial_t \varphi_\varepsilon \\ & \quad - \int_{Q_t} \pi'(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2 + \int_{Q_t} \partial_t \mu_{\mathcal{H}} \partial_t \varphi_\varepsilon. \end{aligned}$$

All the terms on the left-hand side are nonnegative. Moreover, the second inequality in (2.35), the estimates (3.23) and (3.24), and the assumption on μ_Γ in (2.14) combined with (2.39) allow to find a bound for the volume integrals on the right-hand side. Therefore, we deduce that

$$\begin{aligned} & \frac{\tau}{2} \int_\Omega |\partial_t \varphi_\varepsilon(t)|^2 + \int_{Q_t} |\nabla \partial_t \varphi_\varepsilon|^2 \\ & \leq \frac{1}{2} \|\partial_t \varphi_\varepsilon(0)\|_*^2 + \frac{\tau}{2} \int_\Omega |\partial_t \varphi_\varepsilon(0)|^2 + c(\rho + 1) \end{aligned} \quad (3.25)$$

and it remains to estimate the norms of $\partial_t \varphi_\varepsilon(0)$. To this end, we write (3.16) at the time $t = 0$, i.e.,

$$\begin{aligned} & \mathcal{D}(\partial_t \varphi_\varepsilon(0)) + \tau \partial_t \varphi_\varepsilon(0) \\ & = \mathcal{D}((\gamma_1 \sigma_0 - \gamma_2) p(\varphi_0)) + \Delta \varphi_0 - \beta_\varepsilon(\varphi_0) - \rho \text{sign}_\varepsilon(\varphi_0) - \pi(\varphi_0) + \mu_{\mathcal{H}}(0) \end{aligned}$$

and test by $\partial_t \varphi_\varepsilon(0)$. By recalling the first identity in (2.37), (2.35), (3.4) and our assumptions on the data, we easily infer (with the help of the Young inequality) that

$$\|\partial_t \varphi_\varepsilon(0)\|_*^2 + \tau \int_\Omega |\partial_t \varphi_\varepsilon(0)|^2 \leq \frac{\tau}{2} \int_\Omega |\partial_t \varphi_\varepsilon(0)|^2 + \frac{|\Omega|}{\tau} \rho^2 + c.$$

By combining this with (3.25), we deduce that

$$\frac{\tau}{2} \int_\Omega |\partial_t \varphi_\varepsilon(t)|^2 + \int_{Q_t} |\nabla \partial_t \varphi_\varepsilon|^2 \leq \frac{|\Omega|}{\tau} \rho^2 + c(\rho + 1)$$

for a.a. $t \in (0, T)$ and conclude that

$$\|\partial_t \varphi_\varepsilon\|_{L^\infty(0, T; H)} \leq \frac{(2|\Omega|)^{1/2}}{\tau} \rho + C_3(\rho^{1/2} + 1) \quad \text{and} \quad \|\partial_t \varphi_\varepsilon\|_{L^2(0, T; V)} \leq C'_3(\rho + 1). \quad (3.26)$$

Fourth a priori estimate. We write (3.7) in the form

$$-\Delta \mu_\varepsilon = (\gamma_1 \sigma_\varepsilon - \gamma_2) p(\varphi_\varepsilon) - \partial_t \varphi_\varepsilon$$

and owe to (3.26), (3.12) and the Young inequality (using $2^{1/2} < 2$) to deduce that

$$\begin{aligned} \|\Delta\mu_\varepsilon\|_{L^\infty(0,T;H)} &\leq \|\partial_t\varphi_\varepsilon\|_{L^\infty(0,T;H)} + c(\|\sigma_\varepsilon\|_\infty + 1) \\ &\leq \frac{(2|\Omega|)^{1/2}}{\tau} \rho + c(\rho^{1/2} + 1) \leq \frac{2|\Omega|^{1/2}}{\tau} \rho + c. \end{aligned}$$

In particular, we have that (cf. (2.34) and (2.35))

$$\|\mu_\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \leq c(\rho + 1). \quad (3.27)$$

Moreover, we state a more precise L^∞ estimate on account of (2.8). Namely, we have that

$$\|\mu_\varepsilon\|_\infty \leq C_{sh} \frac{2|\Omega|^{2/3}}{\tau} \rho + C_4. \quad (3.28)$$

Fifth a priori estimate. We write (3.8) in the form

$$-\Delta\varphi_\varepsilon + \xi_\varepsilon + \rho\zeta_\varepsilon = \mu_\varepsilon - \tau\partial_t\varphi_\varepsilon - \pi(\varphi_\varepsilon) + \mu_{\mathcal{H}}$$

and multiply by $-\Delta\varphi_\varepsilon$. In dealing with the third term on the left-hand side we split $-\Delta\varphi_\varepsilon$ as $-\Delta(\varphi_\varepsilon - \varphi^*) - \Delta\varphi^*$. By integrating over Ω and rearranging, we infer that

$$\begin{aligned} &\int_\Omega |-\Delta\varphi_\varepsilon|^2 + \int_\Omega \beta'_\varepsilon(\varphi_\varepsilon) |\nabla\varphi_\varepsilon|^2 + \rho \int_\Omega \text{sign}'_\varepsilon(\varphi_\varepsilon - \varphi^*) |\nabla(\varphi_\varepsilon - \varphi^*)|^2 \\ &= \int_\Omega (\mu_\varepsilon - \tau\partial_t\varphi_\varepsilon - \pi(\varphi_\varepsilon) + \mu_{\mathcal{H}}) (-\Delta\varphi_\varepsilon) + \rho \int_\Omega \zeta_\varepsilon \Delta\varphi^* \end{aligned}$$

a.e. in $(0, T)$. We recall that β'_ε and sign'_ε are nonnegative and that $|\zeta_\varepsilon| \leq 1$ a.e. in Q . Thus, using the Young and Schwarz inequalities, we deduce that

$$\frac{1}{2} \int_\Omega |-\Delta\varphi_\varepsilon|^2 \leq \frac{1}{2} \|\mu_\varepsilon - \tau\partial_t\varphi_\varepsilon - \pi(\varphi_\varepsilon) + \mu_{\mathcal{H}}\|_H^2 + \rho |\Omega|^{1/2} \|\Delta\varphi^*\|_H$$

whence

$$\begin{aligned} \|-\Delta\varphi_\varepsilon(t)\|_H &\leq \|(\mu_\varepsilon - \tau\partial_t\varphi_\varepsilon - \pi(\varphi_\varepsilon) + \mu_{\mathcal{H}})(t)\|_H + c\rho^{1/2} \\ &\leq \|\mu_\varepsilon(t)\|_H + \tau\|\partial_t\varphi_\varepsilon(t)\|_H + c(\rho^{1/2} + 1) \quad \text{for a.e. } t \in (0, T) \end{aligned}$$

thanks to (3.24) and the Lipschitz continuity of π . Owing to (3.28) for μ_ε and (3.26) for $\partial_t\varphi_\varepsilon$, we have that

$$\|-\Delta\varphi_\varepsilon\|_{L^\infty(0,T;H)} \leq c\rho + c(\rho^{1/2} + 1) \leq c(\rho + 1).$$

Hence, due to (3.24) and to the regularity theory for elliptic equations, we deduce that

$$\|\varphi_\varepsilon\|_{L^\infty(0,T;W)} \leq C_5(\rho + 1). \quad (3.29)$$

Finally, by comparison we infer that

$$\|\xi_\varepsilon\|_{L^\infty(0,T;H)} \leq C_6(\rho + 1). \quad (3.30)$$

Conclusion. At this point, we collect all the previous estimates and recall that $|\zeta_\varepsilon| \leq 1$ a.e. in Q . Then, using standard compactness results and possibly taking a subsequence, we obtain

$$\varphi_\varepsilon \rightharpoonup \varphi \quad \text{weakly star in } W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \quad (3.31)$$

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{weakly star in } L^\infty(0, T; W_0) \quad (3.32)$$

$$\sigma_\varepsilon \rightharpoonup \sigma \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \quad (3.33)$$

$$\xi_\varepsilon \rightharpoonup \xi \quad \text{weakly star in } L^\infty(0, T; H) \quad (3.34)$$

$$\zeta_\varepsilon \rightharpoonup \zeta \quad \text{weakly star in } L^\infty(Q) \quad (3.35)$$

for some quintuplet $(\varphi, \mu, \sigma, \xi, \zeta)$ satisfying (2.17)–(2.20). Moreover, the estimates we have obtained are conserved for the limiting functions. In particular, those given by (3.28) and (3.12) are conserved, that is, the triplet (φ, μ, σ) satisfies (2.27)–(2.28) with $\widehat{C} = C_4$. Furthermore, in the light of the compact embeddings $H^2(\Omega) \subset V \subset H$ and $H^2(\Omega) \subset C^0(\overline{\Omega})$ and taking advantage, e.g., of [41, Sect. 8, Cor. 4], we derive strong convergence for φ_ε and σ_ε . Namely, we have that

$$\varphi_\varepsilon \rightarrow \varphi \quad \text{strongly in } C^0([0, T]; V) \cap C^0(\overline{Q}) \quad (3.36)$$

$$\sigma_\varepsilon \rightarrow \sigma \quad \text{strongly in } C^0([0, T]; H) \cap L^2(0, T; V). \quad (3.37)$$

We deduce that $\pi(\varphi_\varepsilon)$ and $p(\varphi_\varepsilon)$ converge to $\pi(\varphi)$ and $p(\varphi)$ strongly in $C^0(\overline{Q})$. As $I_\varepsilon(\sigma_\varepsilon) = \sigma_\varepsilon$ for $\varepsilon \leq 1/\sigma^*$, we infer that the products $I_\varepsilon(\sigma_\varepsilon)\pi(\varphi_\varepsilon)$ and $I_\varepsilon(\sigma_\varepsilon)p(\varphi_\varepsilon)$ converge to $\sigma\pi(\varphi)$ and $\sigma p(\varphi)$ strongly in $C^0([0, T]; H)$. Next, by combining (3.36) and (3.34)–(3.35) with well-known monotonicity arguments (see, e.g., [2, Prop. 2.2, p. 38]) we infer that $\xi \in \beta(\varphi)$ and $\zeta \in \text{sign}(\varphi - \varphi^*)$ a.e. in Q . Thus, all the equations (2.21)–(2.24) are satisfied. Finally, on account of the strong convergence in $C^0([0, T]; H)$ for φ_ε and σ_ε , it turns out that (φ, σ) satisfies (2.25) as well. Therefore, the whole problem (2.21)–(2.25) is solved and Theorem 2.2 is completely proved.

4 Existence of sliding modes

This section is devoted to the proof of Theorem 2.3. Our method follows the ideas of [3] and relies on a comparison argument for the equation (2.22). To this end, we define

$$\xi^* := \beta^\circ(\varphi^*) \quad (4.1)$$

and notice that ξ^* is bounded due to (2.15). We also remark that $\varphi_0, \varphi^*, \Delta\varphi^*$ and $\mu_{\mathcal{H}}$ are bounded as well and that $\mu_{\mathcal{H}}$ satisfies $\|\mu_{\mathcal{H}}\|_\infty \leq \|\mu_\Gamma\|_\infty$ (see (2.16), (2.30), (2.26) and (2.38)). Furthermore, we introduce some abbreviations (according to those of Remark 2.4). We set for convenience

$$M := \|\mu_\Gamma\|_\infty + \|\Delta\varphi^*\|_\infty + \|\xi^*\|_\infty, \quad M_0 := \|\varphi_0 - \varphi^*\|_\infty \quad (4.2)$$

$$M_\pi^* := \sup\{|\pi(\varphi^*(x) + r)| : x \in \Omega, |r| \leq M_0\}. \quad (4.3)$$

Next, we recall the definition of C_{sys} given in (2.29) and the estimate (2.27) for μ provided by Theorem 2.2. Thus, we have

$$\|\mu\|_\infty \leq C_{sh} \frac{2|\Omega|^{2/3}}{\tau} \rho + \widehat{C} = C_{sys} \rho + \widehat{C}. \quad (4.4)$$

Finally, we define $A : (0, +\infty) \rightarrow \mathbb{R}$ by setting

$$A(\rho) := C_{sys} \rho + \widehat{C} + M + M_\pi^* \quad \text{for } \rho > 0. \quad (4.5)$$

At this point, we set for convenience

$$\chi := \varphi - \varphi^* \quad (4.6)$$

and write the equation (2.22) in the form

$$\tau \partial_t \chi - \Delta \chi + \xi - \xi^* + \rho \zeta = \mu + \mu_{\mathcal{H}} + \Delta \varphi^* - \xi^* - \pi(\varphi^* + \chi). \quad (4.7)$$

By accounting for the above definitions of M and M_π^* and using (4.4), we obtain the inequalities (which we use later on)

$$\tau \partial_t \chi - \Delta \chi + \xi - \xi^* + \rho \zeta \leq A(\rho) - M_\pi^* - \pi(\varphi^* + \chi) \quad \text{a.e. in } Q \quad (4.8)$$

$$\tau \partial_t \chi - \Delta \chi + \xi - \xi^* + \rho \zeta \geq -A(\rho) + M_\pi^* - \pi(\varphi^* + \chi) \quad \text{a.e. in } Q. \quad (4.9)$$

At this point, we recall that $C_{sys} < 1$ by (2.29) and define ρ^* in this way

$$\rho^* := \frac{1}{1 - C_{sys}} \left(\widehat{C} + M + M_\pi^* + \frac{\tau}{T} M_0 \right) \quad (4.10)$$

so that

$$A(\rho) + \frac{\tau}{T} M_0 < \rho \quad \text{for every } \rho > \rho^*. \quad (4.11)$$

Once such a definition is given, we fix $\rho > \rho^*$ and a solution $(\varphi, \mu, \sigma, \xi, \zeta)$ to problem (2.21)–(2.25) whose component μ satisfies (2.27), i.e., (4.4), and prove the property of the statement. To this end, we notice that (4.11) implies $A(\rho) < \rho$. Thus, the definition

$$T^* := \frac{\tau}{\rho - A(\rho)} M_0 \quad (4.12)$$

makes sense and yields $T^* \in [0, T)$. At this point, we prove that $\chi(t) = 0$, i.e., $\varphi(t) = \varphi^*$, for every $t \in [T^*, T]$ by taking advantage of a comparison argument. We consider the auxiliary problem of finding a pair (w, η) satisfying

$$w \in H^1(0, T), \quad \eta \in L^\infty(0, T) \quad \text{and} \quad \eta \in \text{sign } w \quad \text{a.e. in } (0, T) \quad (4.13)$$

$$\tau w' + \rho \eta = A(\rho) \quad \text{a.e. in } (0, T) \quad \text{and} \quad w(0) = M_0. \quad (4.14)$$

Since sign is a maximal monotone operator, the above problem has a unique solution (w, η) . Moreover, by noticing that $A(\rho)/\rho \in [0, 1) \subset \text{sign } 0$, we see by a simple computation that

$$w(t) = \left(M_0 - \frac{\rho - A(\rho)}{\tau} t \right)^+ \quad \text{for every } t \in [0, T] \quad (4.15)$$

(with $\eta(t) = 1$ and $\eta(t) = A(\rho)/\rho$ for $t < T^*$ and $t > T^*$, respectively). As $w(t) = 0$ for every $t \in [T^*, T]$, it suffices to prove that $|\chi(x, t)| \leq w(t)$ for every $(x, t) \in Q$. To this end, we read w and η as space independent functions defined in Q and the equation in (4.14) as

$$\tau \partial_t w - \Delta w + \rho \eta = A(\rho) \quad \text{a.e. in } Q \quad (4.16)$$

with the boundary condition $\partial_n w = 0$ on Σ . Since $0 \leq w \leq M_0$ by (4.15), we see that $|\pi(\varphi^* \pm w)| \leq M_\pi^*$ by the definition (4.3) of M_π^* . Therefore, w also satisfies the inequalities

$$\tau \partial_t w - \Delta w + \rho \eta \geq A(\rho) - M_\pi^* - \pi(\varphi^* + w) \quad \text{a.e. in } Q \quad (4.17)$$

$$\tau \partial_t w - \Delta w + \rho \eta \geq A(\rho) - M_\pi^* + \pi(\varphi^* - w) \quad \text{a.e. in } Q. \quad (4.18)$$

At this point, we start with the announced comparison argument. We recall the inequalities (4.8) and (4.17) and take the difference. We have

$$\tau \partial_t (\chi - w) - \Delta (\chi - w) + \xi - \xi^* + \rho (\zeta - \eta) \leq \pi(\varphi^* + w) - \pi(\varphi^* + \chi)$$

and multiply this inequality by the positive part $(\chi - w)^+$. By also integrating over Q_t and noticing that $(\chi - w)^+(0) = 0$ by the definition (4.2) of M_0 and the initial condition $w(0) = M_0$, we obtain

$$\begin{aligned} & \frac{\tau}{2} \int_{\Omega} |(\chi - w)^+(t)|^2 + \int_{Q_t} |\nabla (\chi - w)^+|^2 \\ & \quad + \int_{Q_t} (\xi - \xi^*)(\chi - w)^+ + \rho \int_{Q_t} (\zeta - \eta)(\chi - w)^+ \\ & \leq L_\pi \int_{Q_t} |(\chi - w)^+|^2. \end{aligned}$$

Next, we show that the third and fourth term on the left-hand side of the above equality are nonnegative by using the monotonicity of β and sign . Indeed, in the set Q^+ where $(\chi - w)^+ \neq 0$ we have $\chi > w$, whence $\zeta \geq \eta$ since $\zeta \in \text{sign}(\chi)$ and $\eta \in \text{sign}(w)$. In the same set, we also have $\varphi = \varphi^* + \chi > \varphi^* + w \geq \varphi^*$, so that $\xi \geq \xi^*$ since $\xi \in \beta(\varphi)$ and $\xi^* \in \beta(\varphi^*)$. Therefore, we can apply the Gronwall lemma and conclude that $(\chi - w)^+ = 0$ a.e. in Q , i.e., $\chi \leq w$. Now, we consider the sum of the inequalities (4.9) and (4.18), that is,

$$\tau \partial_t (\chi + w) - \Delta (\chi + w) + \xi - \xi^* + \rho (\zeta + \eta) \geq \pi(\varphi^* - w) - \pi(\varphi^* + \chi)$$

and multiply it by $-(\chi + w)^-$. We analogously have that

$$\begin{aligned} & \frac{\tau}{2} \int_{\Omega} |(\chi + w)^-(t)|^2 + \int_{Q_t} |\nabla (\chi + w)^-|^2 \\ & \quad - \int_{Q_t} (\xi - \xi^*)(\chi + w)^- - \rho \int_{Q_t} (\zeta + \eta)(\chi + w)^- \\ & \leq L_\pi \int_{Q_t} |(\chi + w)^-|^2 \end{aligned}$$

and we can treat the integrals involving the nonlinearities similarly as before. In the set Q^- where $(\chi + w)^- \neq 0$ we have $\chi < -w$, whence $\zeta \leq -\eta$ since $\zeta \in \text{sign}(\chi)$ and $-\eta \in \text{sign}(-w)$. In the same set, we also have $\varphi = \varphi^* + \chi < \varphi^* - w \leq \varphi^*$, so that $\xi \leq \xi^*$ since $\xi \in \beta(\varphi)$ and $\xi^* \in \beta(\varphi^*)$. Therefore, we can apply the Gronwall lemma also in this case and deduce that $(\chi + w)^- = 0$ a.e. in Q , i.e., that $-\chi \leq w$. By combining the inequalities we have obtained, we conclude that $|\chi| \leq w$, i.e., that $|\varphi - \varphi^*| \leq w$. Therefore, this and (4.15) imply that $\varphi(t) = \varphi^*$ for $t \geq T^*$, and the proof is complete.

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