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Abstract. In this paper we consider a generalized form of the Discrete Fourier Transform (DFT), called Generalized Discrete Fourier Transform (GFT). Two fast algorithms are given that allow us to obtain a one-dimensional GFT by evaluating a proper multidimensional GFT. It is proved that such two algorithms involve exactly the same arithmetic operations on the same data, and that in the case of the DFT the Algorithm I represents a form of the classical FFT algorithm in mixed radix based on decimation in time. It is shown that the Algorithm II is particularly advantageous for evaluating the DFT on large sets of data.

1. INTRODUCTION

Let us consider the problem of evaluating the one-dimensional Discrete Fourier Transform (DFT) of a vector E having a large number of elements. If the working memory of the available processor is not sufficient to handle the vector as a whole, it is convenient to fracture E into a matrix F^2 and to process separately single columns and single rows [1]. Unfortunately, the one-dimensional DFT of E is not obtainable simply by evaluating the two-dimensional DFT of F^2, but proper "twiddle-factors" must be introduced between column-transforms and row-transforms [2]. Due to the presence of the twiddle factors, such two-dimensional processing, even if each column DFT and each row DFT is evaluated by means of a fast algorithm

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(FFT algorithm), results more complex than an FFT algorithm applied directly to the vector E.

In a previous work [3] the Generalized Discrete Fourier Transform (GFT), that includes the DFT as particular case, has been introduced and fast algorithms for the GFT computation have been given. It has been shown that the one-dimensional DFT of a vector E can be obtained simply by evaluating a two-dimensional GFT of matrix F^2, and that, if each column GFT and each row GFT is computed by means of a GFT fast algorithm, such a two-dimensional processing has the same complexity than an FFT algorithm applied directly to the vector E.

In this work the relations between the fast algorithms for the GFT and the classical FFT algorithms are further investigated, and the following results are obtained.

1) An algorithm (Algorithm I) is given that allows us to obtain a one-dimensional GFT of a vector E by evaluating an α-dimensional GFT of a proper α-dimensional array F^α. It is shown that, in the case of the DFT, this algorithm represents a form of the classical FFT algorithm based on decimation in time. It follows that such classical algorithm can also be viewed as a proper multidimensional GFT.
2) Another algorithm (Algorithm II) is given that consists in rearranging the elements of E into an array F^τ $(\tau < \alpha)$ having arbitrary dimensions, and by computing by means of the Algorithm I successive one-dimensional GFTs along every coordinate of F^τ. It is shown that the Algorithm I and the Algorithm II involve exactly the same arithmetic operations on the same data.

From points 1) and 2) it follows that the computation of the DFT of a vector E by means of the Algorithm II involves exactly the same arithmetic operations on the same data as an FFT algorithm, but presents the advantage of handling a number of elements at a time optimized with respect to the dimensions of the working area.

The proof of the Theorems presented in this paper are omitted for the sake of brevity, and they can be found in [4].

2. GENERALIZED DISCRETE FOURIER TRANSFORM

In this section the one-dimensional GFT and the multidimensional GFT are precisely defined.

Definition 1. Let

$E = \{e_n\}$,

where $n=0,1,\dots,N-1$, and

$G = \{g_k\}$,

where $k=0,1,\dots,N-1$, be two vectors of N complex numbers, and let a and b be two constants. The vector G is said to be the one-dimensional Generalized Discrete Fourier Transform (GFT) of the vector E, with time parameter a and frequency parameter b (briefly, the one-dimensional GFT of (E,a,b)), if

$$g_k = \sum_{n=0}^{N-1} e_n \, W[N]^{(n+a)(k+b)} ,$$

where $W[N]=\exp(-2\pi\sqrt{-1}/N)$.

Note that, as a particular case, the one-dimensional GFT of E with both parameters equal to zero coincides with the one-dimensional DFT of E.

Definition 2. Let

$F^\sigma = \{f^\sigma_{n_1,n_2,\dots,n_\sigma}\}$

$H^\sigma = \{h^\sigma_{k_1,k_2,\dots,k_\sigma}\}$

$$\Phi^\sigma = \{a_1^\sigma=a_1^\sigma(n_2,n_3,\dots,n_\sigma), a_2^\sigma=a_2^\sigma(k_1,n_3,\dots,n_\sigma),\dots,a_\sigma^\sigma=a_\sigma^\sigma(k_1,k_2,\dots,k_{\sigma-1})\}$$

$$\Psi^\sigma = \{b_1^\sigma=b_1^\sigma(n_2,n_3,\dots,n_\sigma), b_2^\sigma=b_2^\sigma(k_1,n_3,\dots,n_\sigma),\dots,b_\sigma^\sigma=b_\sigma^\sigma(k_1,k_2,\dots,k_{\sigma-1})\}$$

(where $n_i,k_i=0,1,\dots,N_i-1$ for $i=1,2,\dots,\sigma$) be, respectively, two σ-dimensional arrays (having the same dimensions) of $\prod_{i=1}^{\sigma} N_i$ complex numbers, and two vectors of σ functions of σ-1 integers. The array H^σ is said to be the <u>σ-dimensional GFT of the array F^σ with time parameter vector Φ^σ and frequency parameter vector Ψ^σ</u> (briefly the σ-dimensional GFT of $(F^\sigma,\Phi^\sigma,\Psi^\sigma)$) if

$$h^\sigma_{k_1,k_2,\ldots,k_\sigma} = \sum_{n_\sigma=0}^{N_\sigma-1}\Bigg(\cdots\Bigg(\sum_{n_2=0}^{N_2-1}\Bigg(\sum_{n_1=0}^{N_1-1} f^\sigma_{n_1,n_2,\ldots,n_\sigma}\, W[N_1]^{(n_1+a_1^\sigma)(k_1+b_1^\sigma)}\Bigg)W[N_2]^{(n_2+a_2^\sigma)(k_2+b_2^\sigma)}\Bigg)\cdots\Bigg) W[N_\sigma]^{(n_\sigma+a_\sigma^\sigma)(k_\sigma+b_\sigma^\sigma)} .$$

Note that the calculation of the σ-dimensional GFT of $(F^\sigma,\Phi^\sigma,\Psi^\sigma)$, i.e. of the array H^σ, can be obtained in σ steps, in the i-th of which, $i=1,2,\ldots,\sigma$, a σ-dimensional array $D[i]^\sigma$ is processed ($D[1]^\sigma=F^\sigma$) and a σ-dimensional array $D[i+1]^\sigma$ is produced, in such a way that $H^\sigma=D[\sigma+1]^\sigma$. To be precise, in the i-th step, for every value combination of $k_1,\ldots,k_{i-1},n_{i+1},\ldots,n_\sigma$, say $k_1^*,\ldots,k_{i-1}^*,n_{i+1}^*,\ldots,n_\sigma^*$, it is evaluated the one-dimensional GFT of the vector

$$\{d[i]^\sigma{}_{k_1^*,\ldots,k_{i-1}^*,n_i,n_{i+1}^*,\ldots,n_\sigma^*}\} , \quad n_i=0,1,\ldots,N_{i-1} ,$$

with parameters

$$a_i^\sigma(k_1^*,\ldots,k_{i-1}^*,n_{i+1}^*,\ldots,n_\sigma^*), \quad b_i^\sigma(k_1^*,\ldots,k_{i-1}^*,n_{i+1}^*,\ldots,n_\sigma^*) ,$$

so obtaining the vector

$$\{d[i+1]^\sigma{}_{k_1^*,\ldots,k_{i-1}^*,k_i,n_{i+1}^*,\ldots,n_\sigma^*}\}, \quad k_i=0,1,\ldots,N_{i-1} ,$$

where $d[\mu]_{k_1,\ldots,k_{\mu-1},n_\mu,\ldots,\mu_\sigma}$ is the $(k_1,\ldots,k_{\mu-1},n_\mu,\ldots,n_\sigma)$-th element of $D[\mu]^\sigma$.

3. TWO FAST ALGORITHMS FOR THE GFT COMPUTATION

From now on the following entities are considered

$$F^{\alpha} = \{f^{\alpha}_{t_1,t_2,\dots,t_\alpha}\}$$

$$H^{\alpha} = \{h^{\alpha}_{z_1,z_2,\dots,z_\alpha}\}$$

$$\Phi^{\alpha} = \{a_1^{\alpha}=a_1^{\alpha}(t_2,t_3,\dots,t_\alpha),a_2^{\alpha}=a_2^{\alpha}(z_1,t_3,\dots,t_\alpha),\dots,a_\alpha^{\alpha}=a_\alpha^{\alpha}(z_1,z_2,\dots,z_{\alpha-1})\}$$

$$\Psi^{\alpha} = \{b_1^{\alpha}=b_1^{\alpha}(t_2,t_3,\dots,t_\alpha),b_2^{\alpha}=b_2^{\alpha}(z_1,t_3,\dots,t_\alpha),\dots,b_\alpha^{\alpha}=b_\alpha^{\alpha}(z_1,z_2,\dots,z_{\alpha-1})\}$$

(where t_s, $z_s=0,1,\dots,T_s-1$ for $s=1,2,\dots,\alpha$) will denote, respectively, two α-dimensional arrays (having the same dimensions) of $\prod_{s=1}^{\alpha} T_s$ complex numbers, and two parameter vectors of α functions of $\alpha-1$ integers.

Likewise

$$F^{\tau} = \{f^{\tau}_{p_1,p_2,\dots,p_\tau}\}$$

$$H^{\tau} = \{h^{\tau}_{q_1,q_2,\dots,q_\tau}\}$$

$$\Phi^{\tau} = \{a_1^{\tau}=a_1^{\tau}(p_2,p_3,\dots,p_\tau),a_2^{\tau}=a_2^{\tau}(q_1,p_3,\dots,p_\tau),\dots,a_\tau^{\tau}=a_\tau^{\tau}(q_1,q_2,\dots,q_{\tau-1})\}$$

$$\Psi^{\tau} = \{b_1^{\tau}=b_1^{\tau}(p_2,p_3,\dots,p_\tau),b_2^{\tau}=b_2^{\tau}(q_1,p_3,\dots,p_\tau),\dots,b_\tau^{\tau}=b_\tau^{\tau}(q_1,q_2,\dots,q_{\tau-1})\}$$

(where p_r, $q_r=0,1,\dots,P_r-1$ for $r=1,2,\dots,\tau$) will denote, respectively, two τ-dimensional arrays (having the same dimensions) of $\prod_{r=1}^{\tau} P_r$ complex numbers, and two parameter vectors of τ functions of $\tau-1$ integers.

Moreover, Δ^{α} and Δ^{τ} will denote, respectively, the sets of integers $\{T_1,T_2,\dots,T_\alpha\}$ and $\{P_1,P_2,\dots,P_\tau\}$.

In the following it is shown how the elements of a one-dimensional GFT of a vector E, with given parameters a and b, can be obtained by regarding the elements of E as reordered in a proper multidimensional array, and by evaluating a multidimensional GFT of such an array with proper parameter vectors.

Definition 3. The array F^{σ} is the Δ^{σ}-horizontal rearrangment of the vector E, where $\Delta^{\sigma}=\{N_1,N_2,\dots,N_{\sigma}\}$, if $N=\prod_{i=1}^{\sigma} N_i$ and

$$\begin{cases} f^{\sigma}_{n_1,n_2,\dots,n_{\sigma}} = e_n \quad \text{for} \\ n = \sum_{u=1}^{\sigma}\left(\prod_{v=u+1}^{\sigma} N_v\right) n_u \end{cases}$$

Likewise, the vector G is the Δ^{σ}-vertical rearrangment of the array H^{σ} if $N=\prod_{i=1}^{\sigma} N_i$ and

$$\begin{cases} g_k = h^{\sigma}_{k_1,k_2,\dots,k_{\sigma}} \quad \text{for} \\ k = \sum_{u=1}^{\sigma}\left(\prod_{v=1}^{u-1} N_v\right) k_u \end{cases}$$

Definition 4. The parameter vectors Φ^{σ} and Ψ^{σ} are the Δ^{σ}-projections of a and b if $N=\prod_{i=1}^{\sigma} N_i$ and

$$\begin{cases} a^{\sigma}_u = \begin{cases} 0 & \text{if } 1 \leqslant u < \sigma \\ a & \text{if } u = \sigma \end{cases} \\ \qquad\qquad\qquad\qquad u=1,2,\dots,\sigma \\ b^{\sigma}_u = \dfrac{\sum_{x=1}^{u-1}\left(\prod_{v=1}^{x-1} N_v\right) k_x + b}{\prod_{x=1}^{u-1} N_x} \end{cases}$$

Theorem 1. Let us consider the vector E and the parameters a and b, and let us suppose that $N=T_1\, T_2 \cdots T_{\alpha}$. The vector G obtained after the following Algorithm I is the one-dimensional GFT of (E, a,b).

Algorithm I on (E,a,b,Δ^{α}).

1) Reorder the elements of E in a α-dimensional array F^{α}, in such a way that F^{α} is the Δ^{α}-horizontal rearrangment of E.
2) Evaluate the α-dimensional GFT of $(F^{\alpha},\Phi^{\alpha},\Psi^{\alpha})$, where Φ^{α} and Ψ^{α} are the Δ^{α}-projections of a and b, so giving a new α-dimensional array H^{α}.
3) Reorder the elements of H^{α} in a vector G, in such a way that G is the Δ^{α}-vertical rearrangment of H^{α}.

The following Theorem links the Algorithm I with the classical FFT algorithm in mixed radix based on decimation in time [5].

Theorem 2. Let us consider the vector E and let us suppose that $N=T_1\ T_2 \cdots T_{\alpha}$. The Algorithm I on $(E,0,0,\Delta^{\alpha})$ represents a form of the classical FFT algorithm in mixed radix based on decimation in time.

From the Algorithm I and from the Definition 2, the following Algorithm II arises.

Theorem 3. Let us consider the vector E and the parameters a and b and let us suppose that $N=P_1\ P_2 \cdots P_{\tau}$ and that $P_r=T_{\beta_r+1}\ T_{\beta_r+2}\cdots T_{\beta_{r+1}}$, $r=1,2,\ldots,\tau$, where $\beta_1=0$, $\beta_{r+1}\geq\beta_r$ and $\beta_{\tau+1}=\alpha$. The vector G obtained after the following Algorithm II is the one-dimensional GFT of (E,a,b).

Algorithm II on $(E,a,b,\Delta^{\tau},\Delta^{\alpha})$.

1) Reorder the elements of E in a τ-dimensional array F^{τ} in such a way that F^{τ} is the Δ^{τ}-horizontal rearrangment of E.
2) Evaluate the τ-dimensional GFT of $(F^{\tau},\Phi^{\tau},\Psi^{\tau})$, where Φ^{τ} and Ψ^{τ} are the Δ^{τ}-projections of a and b, by calculating the one-dimensional GFT of $(\{d[i]_{q_1^*,\ldots,q_{i-1}^*,p_i,p_{i+1}^*,\ldots,p_{\tau}^*}\},a_i^{\tau},b_i^{\tau})$ (see note of Definition 2), by means of the Algorithm I on $(\{d[i]_{q_1^*,\ldots,q_{i-1}^*,p_i,p_{i+1}^*,\ldots,p_{\tau}^*}\},a_i^{\tau},b_i^{\tau},\{T_{\beta_i+1},T_{\beta_i+2},\ldots,T_{\beta_{i+1}}\})$.
3) Reorder the elements of the τ-dimensional array obtained after point 2), say H^{τ}, in a vector G, in such a way that G is the Δ^{τ}-vertical rearrangment of H^{τ}.

The following Theorem establishes an equivalence relation between the Algorithm I and the Algorithm II.

Theorem 4. In the hypothesis of Theorem 2, the Algorithm I on (E,a,b,Δ^{α}) and the Algorithm II on $(E,a,b,\Delta^{\tau},\Delta^{\alpha})$ involve exactly the same arithmetic operations on the same data.

4. CONCLUDING REMARKS

In this work two fast algorithms for computing a one-dimensional GFT are presented, that involve exactly the same aritmetic operations on the same data. Moreover, it is proved that in the case of the DFT of a vector, such algorithms involve exactly the same arithmetic operations on the same data as the classical FFT algorithm in mixed radix based on decimation in time. In order to explain the relevance of the Algorithm II, let us consider the problem of evaluating the DFT of a vector E having a large number N of elements, in the hypothesis that N is a power of T. Moreover, let us assume we have a structure (hardware or software) able to evaluate, by utilizing the Algorithm I, a one-dimensional GFT on a vector having at most P elements, where P is a power of T and N is a power of P. In this case the best solution is to use the Algorithm II that consists in reordering the elements of E in an array F^{τ} having all the dimensions equal to P, and in evaluating by means of the Algorithm I successive one-dimensional GFT (along every coordinate of F^{τ}) of vectors having P elements each.

Note that for the GFT computation there exist also algorithms that, in the case of the DFT, involve exactly the same arithmetic operations on the same data as the classical FFT algorithm in mixed radix based on decimation in frequency. Such algorithms are not presented in this paper, and a more general discussion can be found in [4].

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