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The algorithm, implemented in the finite element code NOSA [4, 5], is used to numerically solve some equilibrium problems whose explicit solutions are known. Successively we study the problem of a masonry spherical dome subjected to its own weight and a point load at the crown which is progressively increased until collapse.

2. The constitutive equation

In this section we prove the main properties of the constitutive equation of masonry-like materials and we determine the eigenvalues of the inelastic deformation and stress as functions of the eigenvalues of the total deformation. Let us state some notation: let $\mathcal{V}$ be a three-dimensional linear space and Lin the space of all linear applications of $\mathcal{V}$ into $\mathcal{V}$, equipped with the inner product

$$\mathbf{A}\cdot\mathbf{B} = \mathrm{tr}\,(\mathbf{A}^T\mathbf{B})\,, \quad \mathbf{A}, \mathbf{B} \in \mathrm{Lin},$$

with $\mathbf{A}^T$ the transpose of $\mathbf{A}$. Let us indicate as Sym, Sym^+ and Sym^- the subsets of Lin constituted by symmetric, symmetric positive semi-definite and symmetric negative semi-definite tensors, respectively. Finally, given an orthonormal basis $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ of $\mathcal{V}$, if $\mathbf{u} \in \mathcal{V}$, $\mathbf{u} \leq \mathbf{0}$ ($\mathbf{u} \geq \mathbf{0}$) means that the components of $\mathbf{u}$ with respect to $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ are non-positive (non-negative).
Let us assume that the tensor of infinitesimal strain $\mathbf{E}$ is the sum of an elastic part $\mathbf{E}^e$ and of an inelastic part $\mathbf{E}^a$, positive semi-definite:

(2.1) $$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^a\,, \quad \mathbf{E}^a \in \mathrm{Sym}^+,$$

and that the Cauchy stress tensor $\mathbf{T}$ depends linearly and isotropically on $\mathbf{E}^e$,

(2.2) $$\mathbf{T} = 2\mu\mathbf{E}^e + \lambda\,\mathrm{tr}(\mathbf{E}^e)\mathbf{I},$$

where the Lame' moduli of the material μ and λ satisfy the inequalities

(2.3) $$\mu > 0, \quad 2\mu + 3\lambda > 0.$$

Moreover, given a non-negative number σ, let us suppose that

(2.4) $$\mathbf{T} - \sigma\,\mathbf{I} \in \mathrm{Sym}^-, \quad (\mathbf{T} - \sigma\,\mathbf{I})\cdot\mathbf{E}^a = 0.$$

Now we prove that, by virtue of (2.1) and (2.4), tensors $\mathbf{T}$ and $\mathbf{E}^a$ are coaxial, a property that will be useful later; to this end it is enough to prove that $\mathbf{T}$ and $\mathbf{E}^a$ commute. In fact because of the symmetry of $\mathbf{T}$ and $\mathbf{E}^a$, there exist two orthonormal bases $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ and $\{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3\}$ of $\mathcal{V}$ such that

$$\mathbf{T} = \sum_{i=1}^{3} t_i\, \mathbf{h}_i \otimes \mathbf{h}_i, \quad \mathbf{E}^a = \sum_{i=1}^{3} a_i\, \mathbf{g}_i \otimes \mathbf{g}_i\,,$$

where t_1, t_2, t_3 and a_1, a_2, a_3 are, respectively, the eigenvalues of $\mathbf{T}$ and $\mathbf{E}^a$ satisfying, in view of $(2.1)_2$ and $(2.4)_1$, the inequalities

(2.5) $$t_i \leq \sigma, \quad a_i \geq 0, \quad i = 1, 2, 3.$$

From $(2.4)_2$ it follows that $\sum_{i,j=1}^{3} (t_j - \sigma) a_i\, (\mathbf{h}_j \cdot \mathbf{g}_i)^2 = 0$ and, thus, in view of (2.5), we have

(2.6) $$(t_j - \sigma) a_i\, \mathbf{h}_j \cdot \mathbf{g}_i = 0\,, \quad i, j = 1, 2, 3.$$

From (2.6) we get

$$\mathbf{T}\,\mathbf{E}^a = \sum_{i,j=1}^{3} (t_j - \sigma) a_i\, (\mathbf{h}_j \cdot \mathbf{g}_i)(\mathbf{h}_j \otimes \mathbf{g}_i) = \mathbf{0};$$

in a similar way we can prove that $\mathbf{E}^a\, \mathbf{T} = \mathbf{0}$, that is $\mathbf{T}$ and $\mathbf{E}^a$ commute, hence they are coaxial.
By virtue of (2.3), relation (2.2) is invertible and we can write

(2.7) $$\mathbf{E}^e = \frac{1}{2\mu}\, \mathbf{T} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\, (\mathrm{tr}\, \mathbf{T})\, \mathbf{I},$$

therefore $\mathbf{T}$ and $\mathbf{E}^e$ are also coaxial.
Now we are in a position to prove that, if conditions (2.3) hold, for every given strain $\mathbf{E}$, there are unique tensors $\mathbf{T}$ and $\mathbf{E}^a$ which satisfy relations (2.1), (2.2) and (2.4). From (2.7) it follows that the relations $(2.1)_1$ and (2.2) are equivalent to

(2.8) $$\mathbf{E} - \frac{\sigma(\mu + \lambda)}{2\mu + 3\lambda}\, \mathbf{I} = \frac{1}{2\mu}\, \mathbf{S} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\, (\mathrm{tr}\, \mathbf{S})\, \mathbf{I} + \mathbf{E}^a,$$

where $\mathbf{S} = \mathbf{T} - \sigma\, \mathbf{I}$. Since tensors $\mathbf{E}$, $\mathbf{E}^a$, $\mathbf{T}$ and $\mathbf{S}$ are coaxial, the constitutive equation $(2.1)_2$, (2.4) and (2.8) can be written with respect to the basis $\{g_1, g_2, g_3\}$ of the eigenvectors of $\mathbf{E}$, $\mathbf{E}^a$ and $\mathbf{T}$. For this purpose, let $\{e_1, e_2, e_3\}$, $\{a_1, a_2, a_3\}$, $\{t_1, t_2, t_3\}$ and $\{s_1, s_2, s_3\}$, with $s_1 = t_1 - \sigma$, $s_2 = t_2 - \sigma$, $s_3 = t_3 - \sigma$, be the eigenvalues of $\mathbf{E}$, $\mathbf{E}^a$, $\mathbf{T}$ and $\mathbf{S}$, respectively. It is easy to prove that the relations (2.1), (2.4) and (2.8) are equivalent to the system

$$
(2.9) \qquad \begin{cases} \tilde{e} = D\,s + a \\ s \leq \mathbf{0} \\ \\ a \geq \mathbf{0} \\ s \cdot a = 0 \end{cases}
$$

where the symmetric matrix

$$
D = \frac{\mu + \lambda}{\mu(2\mu + 3\lambda)} \begin{bmatrix} 1 & -\dfrac{\lambda}{2(\mu+\lambda)} & -\dfrac{\lambda}{2(\mu+\lambda)} \\ -\dfrac{\lambda}{2(\mu+\lambda)} & 1 & -\dfrac{\lambda}{2(\mu+\lambda)} \\ -\dfrac{\lambda}{2(\mu+\lambda)} & -\dfrac{\lambda}{2(\mu+\lambda)} & 1 \end{bmatrix}
$$

is definite positive by virtue of (2.3) and vectors **s**, **a** and $\tilde{e}$ have components

$$
s = (s_1, s_2, s_3), \qquad a = (a_1, a_2, a_3),
$$

$$
\tilde{e} = (e_1 - \frac{\sigma(\mu+\lambda)}{2\mu + 3\lambda}, e_2 - \frac{\sigma(\mu+\lambda)}{2\mu + 3\lambda}, e_3 - \frac{\sigma(\mu+\lambda)}{2\mu + 3\lambda}),
$$

with respect to the basis $\{g_1, g_2, g_3\}$. Now the expected result follows from the fact that the system (2.9) is a linear problem of complementarity, whose solution exists and is unique [6]. Calculation of a_1, a_2, a_3 and t_1, t_2, t_3 as functions of e_1, e_2, e_3 requires definition of the following subsets of Sym:

$$
\mathcal{R}_1 = \{\mathbf{E} \in \mathrm{Sym};\ 2e_1 + \alpha\,(\mathrm{tr}\,\mathbf{E}) - \varepsilon \leq 0,\ 2e_2 + \alpha\,(\mathrm{tr}\,\mathbf{E}) - \varepsilon \leq 0,\ 2e_3 + \alpha\,(\mathrm{tr}\,\mathbf{E}) - \varepsilon \leq 0\},
$$

$$
\mathcal{R}_2 = \{\mathbf{E} \in \mathrm{Sym};\ e_1 \geq \frac{\varepsilon}{2 + 3\alpha}\},
$$

$$
\mathcal{R}_3 = \{\mathbf{E} \in \mathrm{Sym};\ e_1 < \frac{\varepsilon}{2 + 3\alpha},\ \alpha e_1 + 2(1+\alpha)e_2 - \varepsilon \geq 0\},
$$

$$
\mathcal{R}_4 = \{\mathbf{E} \in \mathrm{Sym};\ \alpha e_2 + 2(1+\alpha)e_1 - \varepsilon < 0,\ \alpha e_1 + 2(1+\alpha)e_2 - \varepsilon \leq 0,\ \alpha(\mathrm{tr}\mathbf{E}) + 2e_3 - \varepsilon > 0\},
$$

where $\alpha = \lambda/\mu$, $\varepsilon = \sigma/\mu$ and the eigenvalues e_1, e_2, e_3 are such that $e_1 \leq e_2 \leq e_3$. Since in the applications we are interested in, λ is non-negative, we assume $\alpha \geq 0$. For later use, we observe that from the definition of $\mathcal{R}_3$ and $\mathcal{R}_4$, it clearly follows that in $\mathcal{R}_3$ and $\mathcal{R}_4$ we have $e_1 \neq e_2$ and $e_2 \neq e_3$, respectively.

Solving system (2.9) with a procedure similar to that used in [7], we obtain that the principal components of $\mathbf{E}^a$ can be calculated from the relations:

$$(2.10)_1 \qquad \text{if } \mathbf{E} \in \mathcal{R}_1, \text{ then} \qquad \begin{aligned} a_1 &= 0, \\ a_2 &= 0, \\ a_3 &= 0; \end{aligned}$$

$$(2.10)_2 \qquad \text{if } \mathbf{E} \in \mathcal{R}_2, \text{ then} \qquad \begin{aligned} a_1 &= e_1 - \frac{\varepsilon}{2+3\alpha}, \\ a_2 &= e_2 - \frac{\varepsilon}{2+3\alpha}, \\ a_3 &= e_3 - \frac{\varepsilon}{2+3\alpha}; \end{aligned}$$

$$(2.10)_3 \qquad \text{if } \mathbf{E} \in \mathcal{R}_3, \text{ then} \qquad \begin{aligned} a_1 &= 0, \\ a_2 &= e_2 + \frac{\alpha}{2(1+\alpha)} e_1 - \frac{\varepsilon}{2(1+\alpha)}, \\ a_3 &= e_3 + \frac{\alpha}{2(1+\alpha)} e_1 - \frac{\varepsilon}{2(1+\alpha)}; \end{aligned}$$

$$(2.10)_4 \qquad \text{if } \mathbf{E} \in \mathcal{R}_4, \text{ then} \qquad \begin{aligned} a_1 &= 0, \\ a_2 &= 0, \\ a_3 &= e_3 + \frac{\alpha}{2+\alpha}(e_1 + e_2) - \frac{\varepsilon}{2+\alpha}. \end{aligned}$$

When the principal components of $\mathbf{E}^a$ are known, from $(2.9)_1$ it is possible to calculate the principal stresses t_1, t_2 and t_3. In fact we obtain

$$(2.11)_1 \qquad \text{if } \mathbf{E} \in \mathcal{R}_1, \text{ then} \qquad \begin{aligned} t_1 &= (2\mu + \lambda)e_1 + \lambda\,(e_2 + e_3), \\ t_2 &= (2\mu + \lambda)e_2 + \lambda\,(e_1 + e_3), \\ t_3 &= (2\mu + \lambda)e_3 + \lambda\,(e_2 + e_1); \end{aligned}$$

$$(2.11)_2 \qquad \text{if } \mathbf{E} \in \mathbb{R}_2, \text{ then} \qquad \begin{aligned} t_1 &= \mu\,\varepsilon, \\ t_2 &= \mu\,\varepsilon, \\ t_3 &= \mu\,\varepsilon; \end{aligned}$$

$$(2.11)_3 \qquad \text{if } \mathbf{E} \in \mathbb{R}_3, \text{ then} \qquad \begin{aligned} t_1 &= \frac{\mu}{1+\alpha}\{(2+3\alpha)e_1 + \alpha\,\varepsilon\}, \\ t_2 &= \mu\,\varepsilon, \\ t_3 &= \mu\,\varepsilon; \end{aligned}$$

$$(2.11)_4 \qquad \text{if } \mathbf{E} \in \mathbb{R}_4, \text{ then} \qquad \begin{aligned} t_1 &= \frac{\mu}{2+\alpha}\{4(1+\alpha)e_1 + 2\alpha e_2 + \alpha\,\varepsilon\}, \\ t_2 &= \frac{\mu}{2+\alpha}\{4(1+\alpha)e_2 + 2\alpha e_1 + \alpha\,\varepsilon\}, \\ t_3 &= \mu\,\varepsilon. \end{aligned}$$

Relations (2.10) and (2.12) give the desired solution of the system (2.9).

3. The derivative of T with respect to E

In order to calculate the derivative of $\mathbf{T}$ with respect to $\mathbf{E}$, we have to express $\mathbf{T}$ as function of $\mathbf{E}$, that is to determine the non-linear function $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E})$.

Let us begin by observing that from the coaxiality of $\mathbf{E}$ and $\mathbf{T}$ and the fact that the eigenvalues of $\mathbf{T}$ depend only on the eigenvalues of $\mathbf{E}$, it follows that $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E})$ is an isotropic function. By virtue of a well-known representation theorem, there exist three scalar functions β_0, β_1 and β_2 of invariants of $\mathbf{E}$,

$$I_1(\mathbf{E}) = \operatorname{tr}\mathbf{E} = e_1 + e_2 + e_3,$$

$$I_2(\mathbf{E}) = \mathbf{E}\cdot\mathbf{E} = e_1^2 + e_2^2 + e_3^2,$$

$$I_3(\mathbf{E}) = \mathbf{E}\cdot\mathbf{E}^2 = e_1^3 + e_2^3 + e_3^3$$

such that

$$(3.1) \qquad \mathbf{T} = \beta_0\,\mathbf{I} + \beta_1\,\mathbf{E} + \beta_2\,\mathbf{E}^2.$$

Now we determine β_0, β_1 and β_2 as functions of the eigenvalues of **E** in the four regions $\mathcal{R}_1$, $\mathcal{R}_2$ $\mathcal{R}_3$ and $\mathcal{R}_4$.
In view of (2.11), we have

$$
\text{(3.2)} \quad
\begin{array}{llll}
\text{if } \mathbf{E} \in \mathcal{R}_1, \text{ then} & \beta_0 = \lambda I_1, & \beta_1 = 2\mu, & \beta_2 = 0, \\
\text{if } \mathbf{E} \in \mathcal{R}_2, \text{ then} & \beta_0 = \mu\varepsilon, & \beta_1 = 0, & \beta_2 = 0.
\end{array}
$$

In $\mathcal{R}_3$, the triple $\beta_0, \beta_1, \beta_2$ is a solution of the linear system

$$
\begin{cases}
\beta_0 + \beta_1 e_1 + \beta_2 e_1^2 = \dfrac{\mu}{1+\alpha} \{(2+3\alpha)e_1 + \alpha\varepsilon\} \\
\beta_0 + \beta_1 e_2 + \beta_2 e_2^2 = \mu\varepsilon \\
\beta_0 + \beta_1 e_3 + \beta_2 e_3^2 = \mu\varepsilon,
\end{cases}
$$

therefore, if $\mathbf{E} \in \mathcal{R}_3$, we have

$$
\text{(3.3)} \quad
\begin{aligned}
\beta_0 &= \frac{\kappa}{(e_2 - e_1)(e_3 - e_1)} \{e_1 e_2 e_3 + \\
&\quad \frac{\varepsilon(1+\alpha)}{2+3\alpha} [e_1^2 - e_1 e_2 - e_1 e_3 + \frac{\alpha}{1+\alpha} e_2 e_3]\}, \\
\beta_1 &= \frac{\kappa}{(e_2 - e_1)(e_3 - e_1)} (e_2 + e_3)(\frac{\varepsilon}{2+3\alpha} - e_1), \\
\beta_2 &= \frac{\kappa}{(e_2 - e_1)(e_3 - e_1)} (e_1 - \frac{\varepsilon}{2+3\alpha}),
\end{aligned}
$$

where $\kappa = \dfrac{\mu(2+3\alpha)}{(1+\alpha)}$. Because in $\mathcal{R}_3$ we have $e_1 < e_2 \leq e_3$, the solution given in (3.3) is well-defined. In particular, if $e_2 < e_3$, this solution is also unique.
In $\mathcal{R}_4$ the triple $\beta_0, \beta_1, \beta_2$ is a solution of the linear system

$$
\begin{cases}
\beta_0 + \beta_1 e_1 + \beta_2 e_1^2 = \dfrac{\mu}{2+\alpha} \{4(1+\alpha)e_1 + 2\alpha e_2 + \alpha\varepsilon\} \\
\beta_0 + \beta_1 e_2 + \beta_2 e_2^2 = \dfrac{\mu}{2+\alpha} \{4(1+\alpha)e_2 + 2\alpha e_1 + \alpha\varepsilon\} \\
\beta_0 + \beta_1 e_3 + \beta_2 e_3^2 = \mu\varepsilon,
\end{cases}
$$

therefore, if $\mathbf{E} \in \mathcal{R}_4$, we have

$$
(3.4)\qquad
\begin{aligned}
\beta_0 &= \frac{\xi}{(e_3 - e_1)(e_3 - e_2)} \{-2(2+3\alpha)e_1e_2e_3 + \\
&\quad 2\alpha[e_1e_3(e_3 - e_1) + e_2e_3(e_3 - e_2)] - \\
&\quad \varepsilon[\alpha e_3(e_1 + e_2) - 2(2+\alpha)e_1e_2]\},\\
\beta_1 &= \frac{\xi}{(e_3 - e_1)(e_3 - e_2)} \{\alpha(e_1^2 + e_2^2 + e_3^2) + \\
&\quad 2e_3^2 + (2+3\alpha)e_1e_2 - \varepsilon(e_1 + e_2)\},\\
\beta_2 &= -\frac{\xi}{(e_3 - e_1)(e_3 - e_2)} \{\alpha(e_1 + e_2 + e_3) + 2e_3 - \varepsilon\},
\end{aligned}
$$

where $\xi = \dfrac{2\mu}{2+\alpha}$. Because in $\mathcal{R}_4$ we have $e_1 \leq e_2 < e_3$, the solution given in (3.4) is well-defined. In particular, if $e_1 < e_2$, this solution is also unique.

Now we have to express eigenvalues of **E** as functions of invariantes I_1, I_2 and I_3 of **E**. Since e_1, e_2 and e_3 are the roots of the characteristic polynomial

$$\lambda^3 - I_1(\mathbf{E})\,\lambda^2 + \frac{I_1^2(\mathbf{E}) - I_2(\mathbf{E})}{2}\,\lambda - 2I_3(\mathbf{E}) - I_1^3(\mathbf{E}) + 3I_1(\mathbf{E})\,I_2(\mathbf{E}) = 0,$$

we can write [8]

$$
(3.5)\qquad
\begin{aligned}
e_1 &= -\frac{2}{\sqrt{3}}\chi\cos\omega + \frac{1}{3}I_1,\\
e_2 &= \frac{2}{\sqrt{3}}\chi\cos\left(\omega + \frac{\pi}{3}\right) + \frac{1}{3}I_1,\\
e_3 &= \frac{2}{\sqrt{3}}\chi\cos\left(\omega - \frac{\pi}{3}\right) + \frac{1}{3}I_1,
\end{aligned}
$$

where

$$(3.6)\qquad \cos 3\omega = -\frac{3\sqrt{3}\,\gamma}{2\chi^3},\qquad \gamma = \frac{1}{3}I_3 - \frac{1}{3}I_1I_2 + \frac{2}{27}I_1^3,\qquad \chi = \sqrt{\frac{1}{2}I_2 - \frac{1}{6}I_1^2}.$$

From (3.3) and (3.4), by virtue of (3.5), we obtain the following expressions of β_0, β_1, β_2 for $\mathbf{E} \in \mathcal{R}_3$ and $\mathbf{E} \in \mathcal{R}_4$, respectively:

(3.7)

$$\beta_0 = \frac{\kappa}{\chi^2(4\cos^2\omega - 1)}\{2I_3 + I_1^3 - 3I_1 I_2 + \varepsilon\frac{(1+\alpha)}{2+3\alpha}[\frac{8}{3}\chi^2\cos^2\omega - \frac{2}{3\sqrt{3}}\chi I_1\cos\omega - \frac{I_1^2}{9} + \frac{\alpha}{1+\alpha}(\frac{I_1^2}{9} + \frac{2}{3\sqrt{3}}\chi I_1\cos\omega + \frac{1}{3}\chi^2(4\cos^2\omega - 3))]\},$$

$$\beta_1 = \frac{\kappa}{\chi^2(4\cos^2\omega - 1)}\{(\frac{2}{3}I_1 + \frac{2}{\sqrt{3}}\chi\cos\omega)(\frac{\varepsilon}{2+3\alpha} - \frac{I_1}{3} + \frac{2}{\sqrt{3}}\chi\cos\omega)\},$$

$$\beta_2 = \frac{\kappa}{\chi^2(4\cos^2\omega - 1)}\{\frac{I_1}{3} - \frac{2}{\sqrt{3}}\chi\cos\omega - \frac{\varepsilon}{2+3\alpha}\};$$

(3.8)

$$\beta_0 = \frac{\xi}{4\chi^2\sin\omega(\sin\omega + \sqrt{3}\cos\omega)}\{-2(2+3\alpha)(2I_3 + I_1^3 - 3I_1 I_2) + 2\alpha\chi[\frac{\sqrt{3}}{9}I_1^2(\cos\omega + \sqrt{3}\sin\omega) + \frac{I_1\chi}{3}(2\sqrt{3}\cos\omega\sin\omega - 2\cos^2\omega + 1) - \frac{2}{\sqrt{3}}\chi^2(\cos\omega + \sqrt{3}\sin\omega)] - \varepsilon[-\frac{2}{9}I_1^2 - \frac{4}{3}\chi^2\cos\omega(\sqrt{3}\sin\omega - \cos\omega) + \frac{2}{3\sqrt{3}}I_1\chi(\cos\omega + \sqrt{3}\sin\omega) - 2\alpha\chi^2\sin\omega(\sqrt{3}\cos\omega + \sin\omega)]\},$$

$$\beta_1 = \frac{\xi}{2\chi^2\sin\omega(\sin\omega + \sqrt{3}\cos\omega)}\{\alpha I_2 + \frac{4\chi I_1}{3\sqrt{3}}(\cos\omega + \sqrt{3}\sin\omega) + \frac{2}{9}I_1^2 + \frac{2}{3}\chi^2(3 - 2\cos^2\omega + 2\sqrt{3}\cos\omega\sin\omega) + (2+3\alpha)[\frac{1}{9}I_1^2 - \frac{I_1\chi}{3\sqrt{3}}(\cos\omega + \sqrt{3}\sin\omega) + \frac{2}{3}\chi^2\cos\omega(\sqrt{3}\sin\omega - \cos\omega)] - \varepsilon[\frac{2}{3}I_1 - \frac{\chi}{\sqrt{3}}(\cos\omega + \sqrt{3}\sin\omega)]\},$$

$$\beta_2 = -\frac{\xi}{2\chi^2\sin\omega(\sin\omega + \sqrt{3}\cos\omega)}\{\frac{2+3\alpha}{3}I_1 + \frac{2\chi}{\sqrt{3}}(\cos\omega + \sqrt{3}\sin\omega) - \varepsilon\}.$$

Now, with the help of (3.1), (3.2), (3.7) and (3.8), we are able to calculate the derivative $D_E\mathbf{T}$ of $\mathbf{T}$ with respect to $\mathbf{E}$. Differentiating (3.1) we get

(3.9) $$D_E\mathbf{T} = \mathbf{I}\otimes D_E\beta_0 + \mathbf{E}\otimes D_E\beta_1 + \mathbf{E}^2\otimes D_E\beta_2 + \beta_1\mathbb{1} + \beta_2\mathbb{E},$$

where $\mathbb{1}$ is the fourth-order identity tensor and $\mathbb{E}$ is the fourth-order tensor such that, for $\mathbf{H} \in \mathrm{Sym}$, $\mathbb{E}[\mathbf{H}] = \mathbf{EH} + \mathbf{HE}$.
In particular, in view of (3.2),

$$D_{\mathbf{E}}\mathbf{T} = 2\mu\, \mathbb{1} + \lambda\, \mathbf{I} \otimes \mathbf{I}, \qquad \mathbf{E} \in \mathcal{R}_1 \tag{3.10}$$

and

$$D_{\mathbf{E}}\mathbf{T} = \mathbb{O}, \qquad \mathbf{E} \in \mathcal{R}_2, \tag{3.11}$$

where $\mathbb{O}$ is the fourth-order null tensor.
Calculation of the derivative $D_{\mathbf{E}}\mathbf{T}$ in $\mathcal{R}_3$ and $\mathcal{R}_4$ is performed by observing that β_0, β_1 and β_2 are functions of the invariants I_1, I_2, I_3 and using the expression of their derivatives with respect to $\mathbf{E}$,

$$D_{\mathbf{E}}I_1(\mathbf{E}) = \mathbf{I}, \quad D_{\mathbf{E}}I_2(\mathbf{E}) = 2\mathbf{E}, \quad D_{\mathbf{E}}I_3(\mathbf{E}) = 3\,\mathbf{E}^2. \tag{3.12}$$

In this way, we obtain

$$\begin{aligned} D_{\mathbf{E}}\mathbf{T} = {} & \frac{\partial\beta_0}{\partial I_1}\,\mathbf{I}\otimes\mathbf{I} + 2\,\frac{\partial\beta_0}{\partial I_2}\,\mathbf{I}\otimes\mathbf{E} + \frac{\partial\beta_1}{\partial I_1}\,\mathbf{E}\otimes\mathbf{I} + \\ & 2\,\frac{\partial\beta_1}{\partial I_2}\,\mathbf{E}\otimes\mathbf{E} + \frac{\partial\beta_2}{\partial I_1}\,\mathbf{E}^2\otimes\mathbf{I} + 3\,\frac{\partial\beta_0}{\partial I_3}\,\mathbf{I}\otimes\mathbf{E}^2 + 3\,\frac{\partial\beta_2}{\partial I_3}\,\mathbf{E}^2\otimes\mathbf{E}^2 + \\ & 3\,\frac{\partial\beta_1}{\partial I_3}\,\mathbf{E}\otimes\mathbf{E}^2 + 2\,\frac{\partial\beta_2}{\partial I_2}\,\mathbf{E}^2\otimes\mathbf{E} + \beta_1\,\mathbb{1} + \beta_2\,\mathbb{E}\,. \end{aligned} \tag{3.13}$$

Since the material we are considering is hyperelastic, and its potential $\psi(\mathbf{E})$[1] is a function of class C^2 in the internal part of every region $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ and $\mathcal{R}_4$, the fourth-order tensor $D_{\mathbf{E}}\mathbf{T}$ must be symmetric. Therefore, from (3.13) we obtain

---

[1] In fact, starting from relations (3.1) and (3.2), we can obtain the following expression of the potential

$$\psi(\mathbf{E}) = \begin{cases} \dfrac{\mu}{2}(\alpha I_1^2 + 2I_2), & \mathbf{E} \in \mathcal{R}_1, \\ \dfrac{1}{2}\,\mu\,\varepsilon\, I_1, & \mathbf{E} \in \mathcal{R}_2, \\ \dfrac{1}{2}(\beta_0 I_1 + \beta_1 I_2 + \beta_2 I_3), & \mathbf{E} \in \mathcal{R}_3, \mathcal{R}_4, \end{cases}$$

where β_0, β_1 and β_2 are given in (3.7) and (3.8) for $\mathbf{E}$ belonging to $\mathcal{R}_3$ and $\mathcal{R}_4$, respectively.

(3.14) $$D_E T = \frac{\partial \beta_0}{\partial I_1} I \otimes I + 2 \frac{\partial \beta_0}{\partial I_2} (I \otimes E + E \otimes I) +$$

$$2 \frac{\partial \beta_1}{\partial I_2} E \otimes E + 3 \frac{\partial \beta_0}{\partial I_3} (I \otimes E^2 + E^2 \otimes I) + 3 \frac{\partial \beta_1}{\partial I_3} (E \otimes E^2 + E^2 \otimes E) +$$

$$3 \frac{\partial \beta_2}{\partial I_3} E^2 \otimes E^2 + \beta_1 \mathbb{1} + \beta_2 \mathbb{E}.$$

The expressions of the derivatives $\frac{\partial \beta_0}{\partial I_1}, \frac{\partial \beta_0}{\partial I_2}, \frac{\partial \beta_0}{\partial I_3}, \frac{\partial \beta_1}{\partial I_2}, \frac{\partial \beta_1}{\partial I_3}, \frac{\partial \beta_2}{\partial I_3}$ for $E \in \mathcal{R}_3$ and $\mathcal{R}_4$ are given explicitly in the Appendix.

4. The two-dimensional problem

In this section we calculate the explicit solution of the constitutive equation and the derivative of the stress with respect to the total deformation for plane problems. These calculations are a generalization of the results presented in [1] for materials which do not react to tension. Since many results can be proved in a similar way to that used in [1], their proof is omitted. First of all, we observe that if the eigenvalue $e_3 = g_3 \cdot Eg_3$ of E is zero, or in other words, E is a plane strain, we can prove that the eigenvalue a_3 of E^a is also null and that $t_3 = \frac{\alpha}{2(1+\alpha)}(t_1 + t_2)$, where α is equal to λ/μ, with $\lambda \geq 0$. Let us designate E and E^a as the restrictions of E and E^a to the two-dimensional subspace of $\mathcal{V}$, orthogonal to the vector g_3. For a plane strain, (2.8) becomes $E - \frac{\sigma(2+\alpha)}{2\mu(2+3\alpha)} I = \frac{1}{2\mu} S - \frac{\alpha}{2\mu(2+3\alpha)} (\mathrm{tr}\, S) I + E^a$, and system (2.9) reduces to

(4.1) $$\begin{cases} \tilde{e} = D\, s + a \\ s \leq 0 \\ \\ a \geq 0 \\ s \cdot a = 0 \end{cases}$$

where the matrix

$$D = \frac{1+\alpha}{\mu(2+3\alpha)} \begin{bmatrix} 1 & -\frac{\alpha}{2(1+\alpha)} \\ -\frac{\alpha}{2(1+\alpha)} & 1 \end{bmatrix}$$

is positive definite and

$$s = (s_1, s_2), \qquad a = (a_1, a_2),$$

$$\tilde{e} = (e_1 - \frac{\sigma(2+\alpha)}{2\mu(2+3\alpha)}, e_2 - \frac{\sigma(2+\alpha)}{2\mu(2+3\alpha)}).$$

The calculation of a_1 and a_2 which satisfy (4.1) requires the definition of the following subsets of Sym:

$$\mathcal{S}_1 = \{E \in \text{Sym};\ \alpha e_2 + (2+\alpha)e_1 - \varepsilon \leq 0\ ,\ \ \alpha e_1 + (2+\alpha)e_2 - \varepsilon \leq 0\ \},$$

$$\mathcal{S}_2 = \{\ E \in \text{Sym};\ e_1 \geq \frac{\varepsilon}{2(1+\alpha)},\ e_2 > \frac{\varepsilon}{2(1+\alpha)}\ \},$$

$$\mathcal{S}_3 = \{\ E \in \text{Sym};\ e_1 < \frac{\varepsilon}{2(1+\alpha)},\ \alpha e_1 + (2+\alpha) e_2 - \varepsilon > 0\ \},$$

where we suppose the eigenvalues e_1 and e_2 are ordered in such a way that $e_1 \leq e_2$, and we set $\varepsilon = \sigma/\mu$.
The principal components of E^a can be calculated from the relations

$(4.2)_1$ if $E \in \mathcal{S}_1$, then $a_1 = 0,\qquad a_2 = 0;$

$(4.2)_2$ if $E \in \mathcal{S}_2$, then $a_1 = e_1 - \frac{\varepsilon}{2(1+\alpha)},\qquad a_2 = e_2 - \frac{\varepsilon}{2(1+\alpha)};$

$(4.2)_3$ if $E \in \mathcal{S}_3$, then $a_1 = 0,\qquad a_2 = e_2 + \frac{\alpha}{2+\alpha} e_1 - \frac{\varepsilon}{2+\alpha}.$

By virtue of $(4.1)_1$ and (4.2), the principal components t_1 and t_2 of stress tensor $\mathbf{T}^{(2)}$ are:

$(4.3)_1$ if $\mathbf{E} \in \mathcal{S}_1$, then $t_1 = \mu\{2e_1 + \alpha(e_1 + e_2)\},\qquad t_2 = \mu\{2e_2 + \alpha(e_1 + e_2)\},$

$(4.3)_2$ if $\mathbf{E} \in \mathcal{S}_2$, then $t_1 = \mu\, \varepsilon,\qquad t_2 = \mu\, \varepsilon,$

$(4.3)_3$ if $\mathbf{E} \in \mathcal{S}_3$, then $t_1 = \varphi e_1 + \frac{\mu\varepsilon\alpha}{2+\alpha},\qquad t_2 = \mu\, \varepsilon,$

where $\varphi = \frac{4\mu(1+\alpha)}{(2+\alpha)}$.
Because of the isotropy of the non-linear function $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{E})$, there exist two scalar functions β_0 and β_1 of invariants of $\mathbf{E}$,

$$I_1(\mathbf{E}) = \text{tr}\, \mathbf{E} = e_1 + e_2,\ \ I_2(\mathbf{E}) = \mathbf{E} \cdot \mathbf{E}\,,$$

---

2 As for $\mathbf{E}$ and $\mathbf{E}^a$, $\mathbf{T}$ is used to indicate the restriction of the stress tensor to the two-dimensional subspace of $\mathcal{V}$ orthogonal to $\mathbf{g}_3$.

such that

(4.4) $\quad \mathbf{T} = \beta_0 \mathbf{I} + \beta_1 \mathbf{E}.$

In view of (4.3), we can prove [1] that:

$(4.5)_1$ if $\mathbf{E} \in \mathcal{S}_1$, then $\beta_0 = \lambda I_1, \qquad \beta_1 = 2\mu;$

$(4.5)_2$ if $\mathbf{E} \in \mathcal{S}_2$, then $\beta_0 = \mu\varepsilon, \qquad \beta_1 = 0;$

$(4.5)_3$ if $\mathbf{E} \in \mathcal{S}_3$, then

$$\beta_0 = \frac{\varphi}{2}\,\frac{I_1^2 - I_2}{\sqrt{2I_2 - I_1^2}} + \frac{\varphi\varepsilon}{4(1+\alpha)}\left(\alpha + 1 - \frac{I_1}{\sqrt{2I_2 - I_1^2}}\right),$$

$$\beta_1 = -\,\varphi\,\frac{I_1 - \sqrt{2I_2 - I_1^2}}{2\sqrt{2I_2 - I_1^2}} + \frac{\varphi\varepsilon}{2(1+\alpha)\sqrt{2I_2 - I_1^2}}.$$

Now, differentiating (4.4) with respect to $\mathbf{E}$, we obtain the derivative of $\mathbf{T}$ in the three regions $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ [1].

$(4.6)_1$ If $\mathbf{E} \in \mathcal{S}_1$ $\quad D_{\mathbf{E}}\mathbf{T} = 2\mu\,\mathbb{1} + \lambda\,\mathbf{I} \otimes \mathbf{I};$

$(4.6)_2$ if $\mathbf{E} \in \mathcal{S}_2$ $\quad D_{\mathbf{E}}\mathbf{T} = \mathbb{O};$

$(4.6)_3$ if $\mathbf{E} \in \mathcal{S}_3$

$$D_{\mathbf{E}}\mathbf{T} = \frac{\varphi}{2}\,\frac{I_1(3I_2 - I_1^2) - \dfrac{\varepsilon}{(1+\alpha)}\,I_2}{(2I_2 - I_1^2)^{3/2}}\,\mathbf{I}\otimes\mathbf{I} +$$

$$\varphi\,\frac{-I_2 + \dfrac{\varepsilon}{2(1+\alpha)}\,I_1}{(2I_2 - I_1^2)^{3/2}}\,(\mathbf{I}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{I}) +$$

$$\varphi\,\frac{I_1 - \dfrac{\varepsilon}{1+\alpha}}{(2I_2 - I_1^2)^{3/2}}\,\mathbf{E}\otimes\mathbf{E} + \varphi\,\frac{-I_1 + \sqrt{2I_2 - I_1^2} + \dfrac{\varepsilon}{1+\alpha}}{2\sqrt{2I_2 - I_1^2}}\,\mathbb{1}.$$

Let us suppose that $t_3 = \mathbf{g}_3 \cdot \mathbf{T}\mathbf{g}_3 = 0$, that is to say $\mathbf{T}$ is a plane stress. In view of (2.1) and (2.2) we have

(4.7) $\quad e_3 - a_3 = \dfrac{\alpha}{2+\alpha}\,(a_1 + a_2 - e_1 - e_2);$

moreover a_3 is equal to zero, by virtue of $(2.4)_2$. From (2.2) and (4.7), it follows that

(4.8) $$\mathbf{T} = 2\mu\{(\mathbf{E} - \mathbf{E}^a) + \frac{\alpha}{2+\alpha}\, \mathrm{tr}(\mathbf{E} - \mathbf{E}^a)\mathbf{I}\}\ .$$

Let us define the following subsets of Sym:

$$\mathcal{T}_1 = \{\mathbf{E} \in \mathrm{Sym};\ 2\alpha e_2 + 4(1+\alpha)e_1 - \varepsilon(2+\alpha) \le 0\ ,\ 2\alpha e_1 + 4(1+\alpha)e_2 - \varepsilon(2+\alpha) \le 0\},$$

$$\mathcal{T}_2 = \{\ \mathbf{E} \in \mathrm{Sym};\ e_1 \ge \varepsilon\, \frac{(2+\alpha)^2}{6\alpha^2 + 16\alpha + 8},\ \ e_2 > \varepsilon\, \frac{(2+\alpha)^2}{6\alpha^2 + 16\alpha + 8}\ \},$$

$$\mathcal{T}_3 = \{\ \mathbf{E} \in \mathrm{Sym};\ e_1 < \varepsilon\, \frac{(2+\alpha)^2}{6\alpha^2 + 16\alpha + 8},\ \ 2\alpha e_1 + 4(1+\alpha)e_2 - \varepsilon(2+\alpha) > 0\},$$

with $e_1 \le e_2$.
Also in this case we can write $\mathbf{T} = \beta_0\,\mathbf{I} + \beta_1\,\mathbf{E}$, where

$(4.9)_1$ if $\mathbf{E} \in \mathcal{T}_1$, then $$\beta_0 = \frac{2\lambda}{2+\alpha}\, I_1, \qquad \beta_1 = 2\mu;$$

$(4.9)_2$ if $\mathbf{E} \in \mathcal{T}_2$, then $$\beta_0 = \mu\varepsilon\ , \qquad \beta_1 = 0;$$

$(4.9)_3$ if $\mathbf{E} \in \mathcal{T}_3$, then
$$\beta_0 = \frac{\varphi_1 (I_1^2 - I_2)}{2\sqrt{2I_2 - I_1^2}} + \frac{\varphi_1 \varepsilon}{2(2+3\alpha)} \left(\frac{3\alpha+2}{2} - \frac{(2+\alpha) I_1}{2\sqrt{2I_2 - I_1^2}}\right),$$
$$\beta_1 = -\ \varphi_1\, \frac{I_1 - \sqrt{2I_2 - I_1^2}}{2\sqrt{2I_2 - I_1^2}} + \frac{\varphi_1 \varepsilon (2+\alpha)}{2(2+3\alpha)\sqrt{2I_2 - I_1^2}},$$

with $\varphi_1 = \dfrac{\mu(2+3\alpha)}{1+\alpha}$.
The derivative of $\mathbf{T}$ is

$(4.10)_1$ if $\mathbf{E} \in \mathcal{T}_1$, $$D_E\mathbf{T} = 2\mu\, \mathbb{1} + \frac{2\lambda}{2+\alpha}\, \mathbf{I} \otimes \mathbf{I},$$

$(4.10)_2$ if $\mathbf{E} \in \mathcal{T}_2$, $$D_E\mathbf{T} = \mathbb{O},$$

$(4.10)_3$ if $\mathbf{E} \in \mathcal{T}_3$, $$D_E\mathbf{T} = \frac{\varphi_1}{2}\, \frac{I_1(3I_2 - I_1^2) - \dfrac{\varepsilon(2+\alpha)}{2(2+3\alpha)}\, I_2)}{(2I_2 - I_1^2)^{3/2}}\, \mathbf{I} \otimes \mathbf{I}\ +$$

$$\varphi_1\, \frac{-\ I_2 + \dfrac{\varepsilon(2+\alpha)}{2(2+3\alpha)}\, I_1}{(2I_2 - I_1^2)^{3/2}}\, (\mathbf{I} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{I}) +$$

$$\varphi_1 \frac{I_1 - \dfrac{\varepsilon(2+\alpha)}{2+3\alpha}}{(2I_2 - I_1^2)^{3/2}} \mathbf{E} \otimes \mathbf{E} + \varphi_1 \frac{-I_1 + \sqrt{2I_2 - I_1^2} + \dfrac{\varepsilon(2+\alpha)}{2+3\alpha}}{2\sqrt{2I_2 - I_1^2}} \mathbb{1}.$$

We remark that for $\sigma = 0$, the expressions of $D_E\mathbf{T}$ given in (4.6) and (4.10) reduce to the expressions calculated in [1] for materials which do not support tension.

5. Description of the algorithm

The algorithm described here for the reader's convenience, is wholly analogous to that presented in [1], with the only difference that, here, the possibility of incrementally assigning the load is considered. This possibility has revealed itself essential to the study of the masonry dome analyzed in example 3 of the next section.
The matrix of the engineering components of the tensor $D_E\mathbf{T}$, necessary for calculation of the tangent matrix, is explicitly calculated in the Appendix for three-dimensional and two-dimensional problems. Now, we briefly recall the description of the algorithm implemented in NOSA.
Let us consider the following quantities related to the i-th iteration of the j-th load increment:

$\mathbf{u}^{(i,j)}$	vector of nodal displacements,
$D(\mathbf{u}^{(i,j)})$	matrix of the engineering components of $D_E\mathbf{T}$ (see Appendix),
$K_T(\mathbf{u}^{(i,j)})$	tangent stiffness matrix,
$\mathbf{f}^{(i,j)}$	nodal equivalent of the assigned incremental loads if $i = 0$; nodal equivalent of the residual loads if $i \geq 1$,
$\mathbf{e}_G^{(i,j)}$	vector of the engineering components of the total strain,
$\mathbf{a}_G^{(i,j)}$	vector of the engineering components of inelastic strain ,
$\mathbf{t}_G^{(i,j)}$	vector of the engineering components of stress,

where the subscript G indicates the Gauss point at which these quantities are calculated.
In the first iteration of the first load increment, $\mathbf{u}^{(0,1)}$ is null and $D(\mathbf{u}^{(0,1)})$ coincides with the matrix of elastic moduli. Let us suppose that, during the j-th load increment, we have calculated the displacement $\mathbf{u}^{(i,j)}$, the tangent stiffness matrix $K_T(\mathbf{u}^{(i,j)})$ and the nodal equivalent loads $\mathbf{f}^{(i,j)}$ corresponding to the i - th iteration; we solve the linear system

$$(5.1) \quad K_T(\mathbf{u}^{(i,j)})\, \Delta\mathbf{u}^{(i,j)} = \mathbf{f}^{(i,j)} ,$$

in order to determine the displacement $\mathbf{u}^{(i+1,j)} = \mathbf{u}^{(i,j)} + \Delta\mathbf{u}^{(i,j)}$ relative to the $(i + 1)$ - th iteration.
Then for every Gauss point of every element we calculate the total strain $\mathbf{e}_G^{(i+1,j)}$ associated with the displacement $\mathbf{u}^{(i+1,j)}$, and its eigenvalues which are needed to calculate the inelastic strain $\mathbf{a}_G^{(i+1,j)}$; then we calculate the stress $\mathbf{t}_G^{(i+1,j)}$ using the constitutive relation (2.2). We

observe that the stress is negative semi-definite because it is calculated by directly solving the constitutive equation (2.1), (2.2), (2.4).
Moreover, using relations (A.1) for three-dimensional problems and (A.3) for two-dimensional problems, we arrive at matrix $D(\mathbf{u}^{(i+1,j)})$ which, if necessary, may be used in the next iteration or in the next load increment.
Finally, we calculate the vector of residual loads $\mathbf{f}^{(i+1,j)}$ and perform the convergence check

$$(5.2) \qquad \frac{|\mathbf{f}^{(i+1,j)}|}{\sum_{k=1}^{j} |\mathbf{f}^{(0,k)}|} \leq \xi_c ;$$

if convergence has not been reached, we repeat all operations beginning with the solution of system (5.1).

6. Examples

In this Section we numerically solve two equilibrium problems and compare the results obtained with their corresponding exact solutions. Successively, in example 3, we consider a masonry spherical dome subjected to its own weight and to a point load applied on the keystone; for this case we have no exact solution. In the following, ν is the Poisson ratio and E the Young modulus.

Example 1. *Spherical container subjected to uniform radial loads.*
Let us consider a spherical coordinate system $\{O, \rho, \theta, \varphi\}$ and a spherical container whose centre coincides with O, with inner radius a and outer radius b, subjected to uniform pressures p_e and p_i acting respectively on the outer and inner surface [9]. We solve the equilibrium problem for $\sigma = 0$, namely we consider a material not supporting tension. If ratio p_e/p_i belongs to the interval $\left[\frac{a^2}{b^2}, \frac{2a^3+b^3}{3\,b^3}\right]$, then a unique real root ρ_0 belonging to [a, b] exists for the equation

$$(6.1) \qquad 2p_i\, a^2 \rho_0^3 - 3b^3 p_e\, \rho_0^2 + p_i\, a^2 b^3 = 0,$$

such that ρ_0 separates the region in which the inelastic deformation $\mathbf{E}^a$ is different from $\mathbf{0}$, from the one in which $\mathbf{E}^a$ is null.
The stress field $\mathbf{T}$, negative semi-definite and in equilibrium with pressures p_e and p_i, has spherical components

$$(6.2)_1 \qquad \sigma_\rho(\rho) = \begin{cases} -\dfrac{a^2}{\rho^2}\, p_i\,, & \rho \in [a, \rho_0], \\[2ex] -p_i\left(\dfrac{a^2\rho_0}{3\rho^3} + \dfrac{a^2}{3\rho_0^2}\right), & \rho \in [\rho_0, b]; \end{cases}$$

$$(6.2)_2 \qquad \sigma_\theta(\rho) = \begin{cases} 0, & \rho \in [a, \rho_0], \\ p_i \left(\dfrac{a^2 \rho_0}{3\rho^3} - \dfrac{a^2}{3\rho_0^2} \right), & \rho \in [\rho_0, b]; \end{cases}$$

$$(6.2)_3 \qquad \sigma_\varphi(\rho) = \sigma_\theta(\rho), \ \tau_{\rho\theta}(\rho) = \tau_{\varphi\theta}(\rho) = \tau_{\rho\varphi}(\rho) = 0.$$

The radial displacement u is

$$(6.3) \qquad u(\rho) = \begin{cases} \dfrac{a^2 p_i}{E} \left(\dfrac{1}{\rho} - \dfrac{(1-\nu)}{\rho_0} \right), & \rho \in [a, \rho_0], \\ \dfrac{p_i}{3E} \dfrac{a^2}{\rho_0^2} \left((1+\nu) \dfrac{\rho_0^3}{\rho^2} + (2\nu - 1)\rho \right), & \rho \in [\rho_0, b]; \end{cases}$$

and the circumferential inelastic strain is

$$(6.4) \qquad \varepsilon_\theta^a(\rho) = \varepsilon_\varphi^a(\rho) = \begin{cases} \dfrac{1-\nu}{E} \dfrac{a^2}{\rho} p_i \left(\dfrac{1}{\rho} - \dfrac{1}{\rho_0} \right), & \rho \in [a, \rho_0], \\ 0, & \rho \in [\rho_0, b]. \end{cases}$$

The numerical analysis has been performed using the following values of constants:

$a = 1000$ cm,	$b = 2000$ cm,
$p_i = 1$ Kg/cm^2,	$p_e = 0.305572$ Kg/cm^2,
$\nu = 0.1$,	$E = 50000$ Kg/cm^2.

Using these values, from (6.1), we obtain $\rho_0 = 1300$ cm.

In the finite element analysis, for reasons of symmetry, only a quarter of the spherical container was studied and it was discretized into twenty-seven brick elements with twenty nodes and twenty-seven Gauss points. The tolerance ξ_c is equal to 10^{-6}, the convergence was reached in five iterations and the norm of residual forces is equal to $0.26\ 10^{-9}\ |f^{(0)}|$. Figures 1, 2, 3 and 4 show the radial stress, the circumferential stress, the circumferential inelastic strain and the radial displacement; the continuous line represents the exact solution, the bold points the numerical solution.

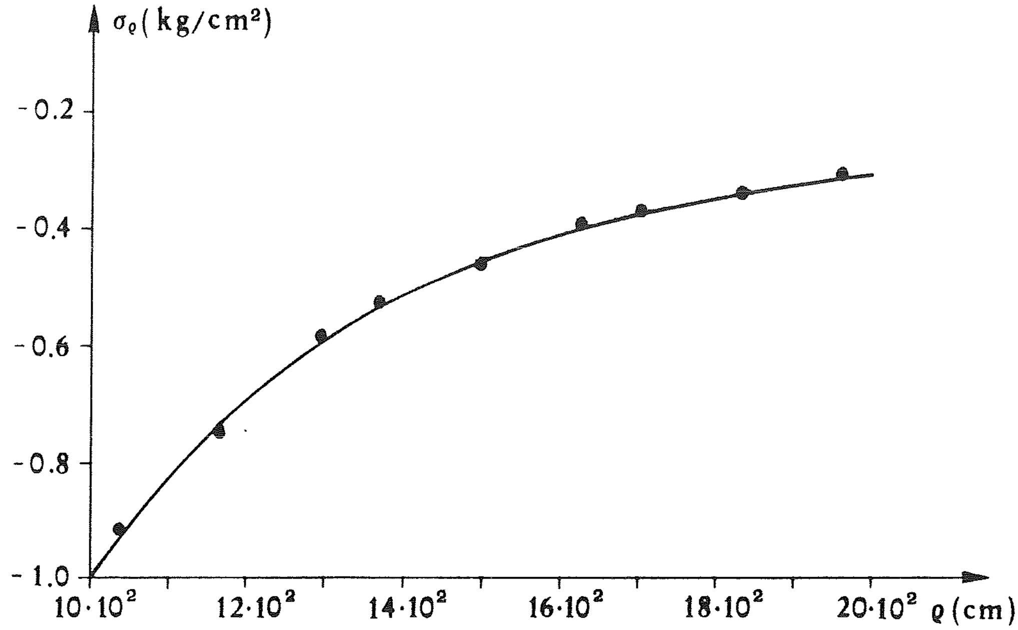

Figure 1. Radial stress vs. ρ.

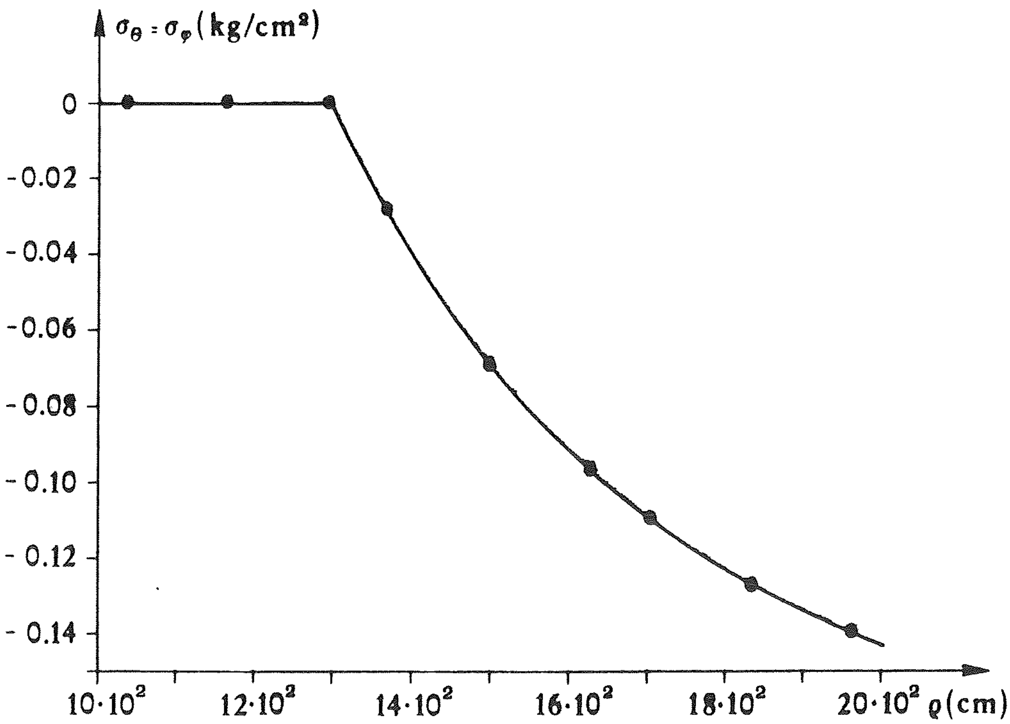

Figure 2. Circumferential stress vs. ρ.

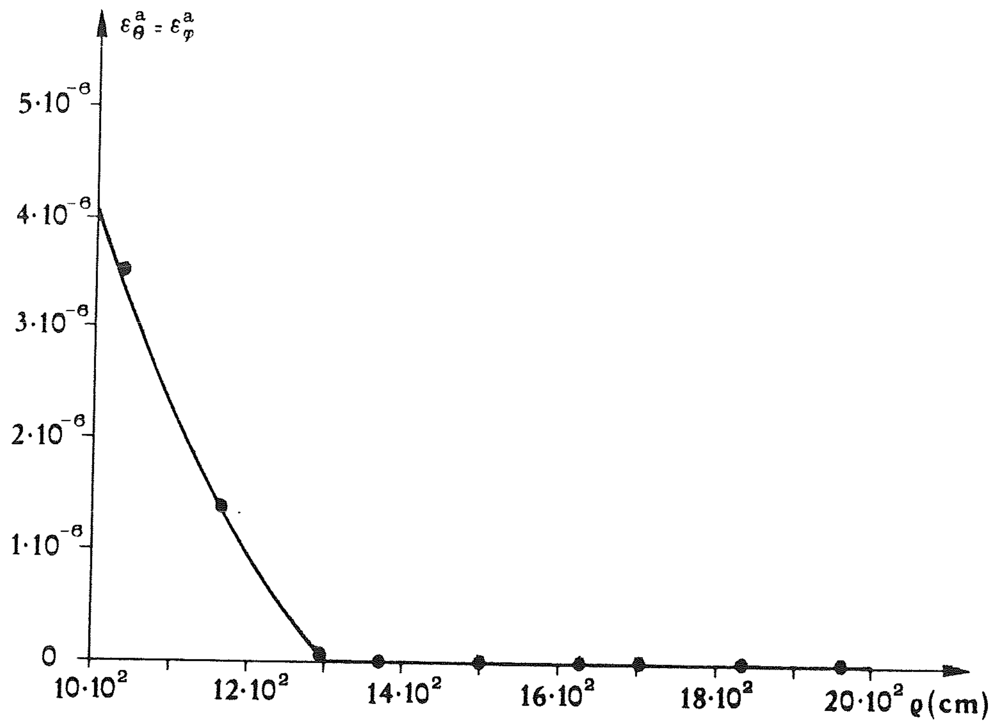

Figure 3. Circumferential inelastic strain vs. ρ.

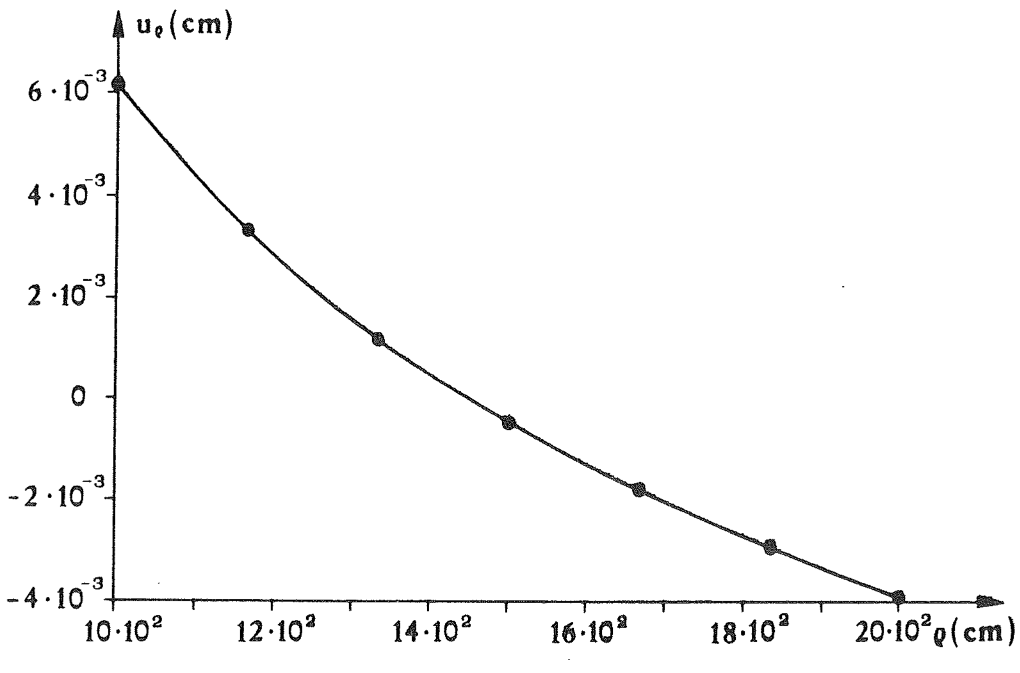

Figure 4. Radial displacement vs. ρ.

Example 2. *Cylindrical container subjected to uniform radial loads and its own weight.*
In a cylindrical reference system $\{O, \rho, \theta, z\}$, let us consider the cylindrical container Ω having inner radius a, outer radius b and height h, subjected to the radial pressures p_e and p_i acting, respectively, on the outer and inner boundary and to its own weight $-p\,\mathbf{j}$. The cylinder is simply supported, that is the vertical displacement $w(\rho, 0)$ is equal to zero. Moreover, let us suppose that the Poisson ratio ν is equal to zero and that the material is able to support a tension σ different fron zero. If the condition $\frac{a}{b} \leq \frac{p_e+\sigma}{p_i+\sigma} \leq \frac{a^2+b^2}{2\,b^2}$ is satisfied, then the stress field $\mathbf{T}$, in equilibrium with the loads $p_e\ p_i$ and $-p\,\mathbf{j}$, such that $\mathbf{T} - \sigma\,\mathbf{I}$ is negative semi-definite, has the principal components

(6.5)
$$\sigma_\rho(\rho) = \begin{cases} \sigma - \frac{a}{\rho}\,(p_i+\sigma), & \rho \in [a, \rho_0], \\ -(p_i+\sigma)\left(\frac{a\,\rho_0}{2\rho^2} + \frac{a}{2\rho_0}\right) + \sigma, & \rho \in [\rho_0, b]; \end{cases}$$

$$\sigma_\theta(\rho) = \begin{cases} \sigma, & \rho \in [a, \rho_0], \\ (p_i+\sigma)\left(\frac{a\,\rho_0}{2\rho^2} - \frac{a}{2\rho_0}\right) + \sigma, & \rho \in [\rho_0, b]; \end{cases}$$

$$\sigma_z(z) = -p\,(h-z), \qquad z \in [0, h];$$

where

(6.6)
$$\rho_0 = \frac{b}{a}\,\frac{b(p_e+\sigma) - \sqrt{b^2(p_e+\sigma)^2 - a^2(p_i+\sigma)^2}}{p_i+\sigma}$$

is the transition radius from the region in which $\mathbf{E}^a \neq \mathbf{0}$ to the one in which $\mathbf{E}^a = \mathbf{0}$.
The radial and vertical displacements u and w and the circumferential component of the inelastic strain are univocally determined and have the expressions

(6.7)
$$u(\rho) = \begin{cases} \frac{1}{E}\left\{a(p_i+\sigma)\ln\left(\frac{\rho_0}{\rho}\right) + \sigma\,\rho\right\}, & \rho \in [a, \rho_0], \\ \frac{1}{E}\left\{\sigma\rho - \frac{a}{2}(p_i+\sigma)\left(\frac{\rho}{\rho_0} - \frac{\rho_0}{\rho}\right)\right\}, & \rho \in [\rho_0, b]; \end{cases}$$

(6.8)
$$w(z) = -\frac{pz}{2E}\,(2h - z) \qquad z \in [0, h];$$

$$(6.9)\quad \varepsilon_\theta^a(\rho) = \begin{cases} \dfrac{a}{\rho E}\,(p_i + \sigma)\ln\left(\dfrac{\rho_0}{\rho}\right), & \rho \in [a, \rho_0], \\ 0, & \rho \in [\rho_0, b]. \end{cases}$$

For the numerical calculation of the solution, the following values of the constants have been used:

$a = 100$ cm,	$b = 200$ cm,	$h = 200$ cm,
$p_i = 10$ Kg/cm^2,	$p_e = 5.5$ Kg/cm^2,	
$p = 0.002$ Kg/cm^3,	$\sigma = 0.5$ Kg/cm^2,	
$\nu = 0.$,	$E = 50000$ Kg/cm^2.	

With these values the transition radius ρ_0 is approximately 117.9 cm. In the finite element analysis fifty axisymmetric elements with eight nodes and nine Gauss points have been used; the tolerance ξ_c is equal to 10^{-5}, the convergence was reached in three iterations and the norm of residual forces is equal to $0.68\ 10^{-12}\,|f^{(0)}|$.

Figures 5, 6, 7 and 8 show the behaviour of the radial stress, circumferential stress, circumferential inelastic strain and radial displacement, none of which depends on z. Figures 9 and 10 show the behaviour of the axial displacement w and the axial stress, which also do not depend on ρ.

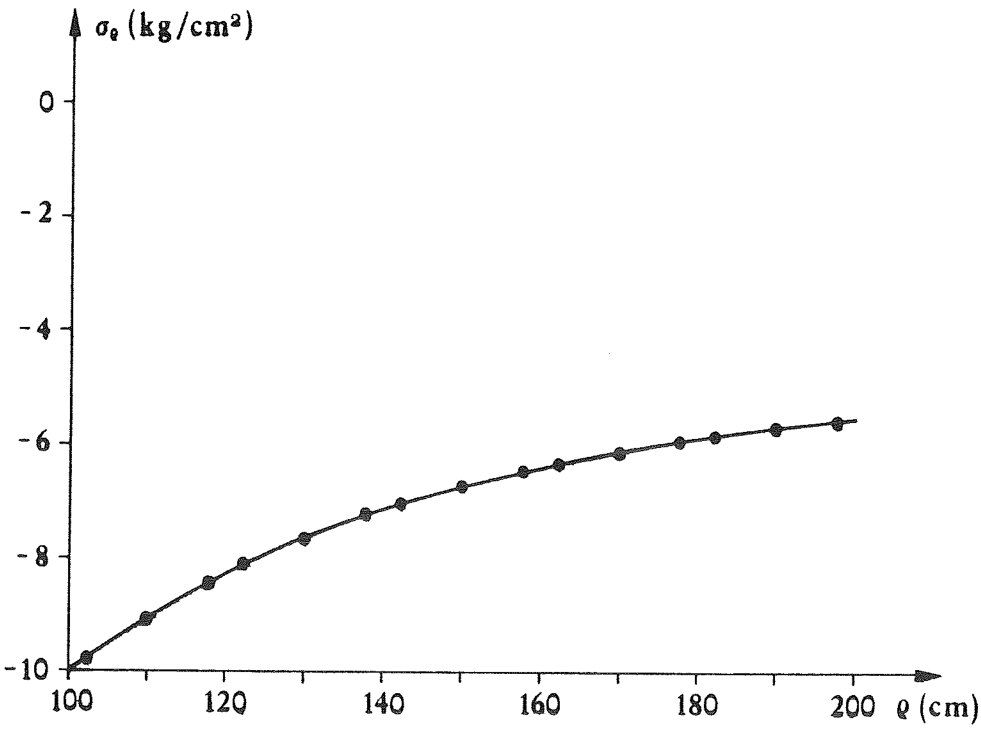

Figure 5. Radial stress vs. ρ.

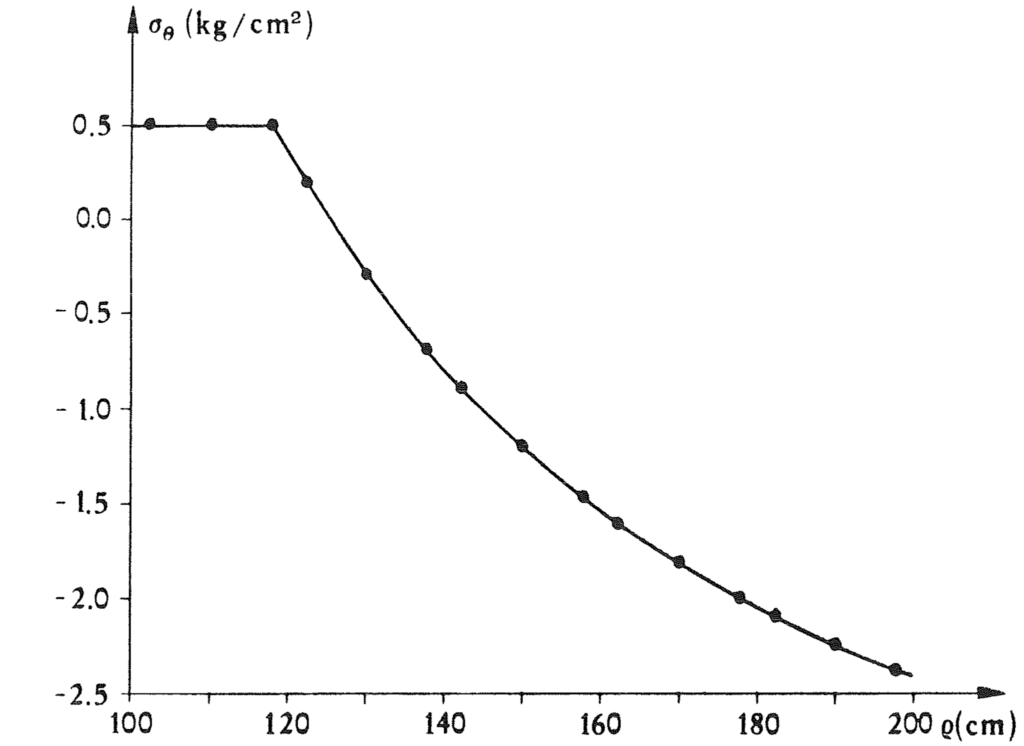

Figure 6. Circumferential stress vs. ρ.

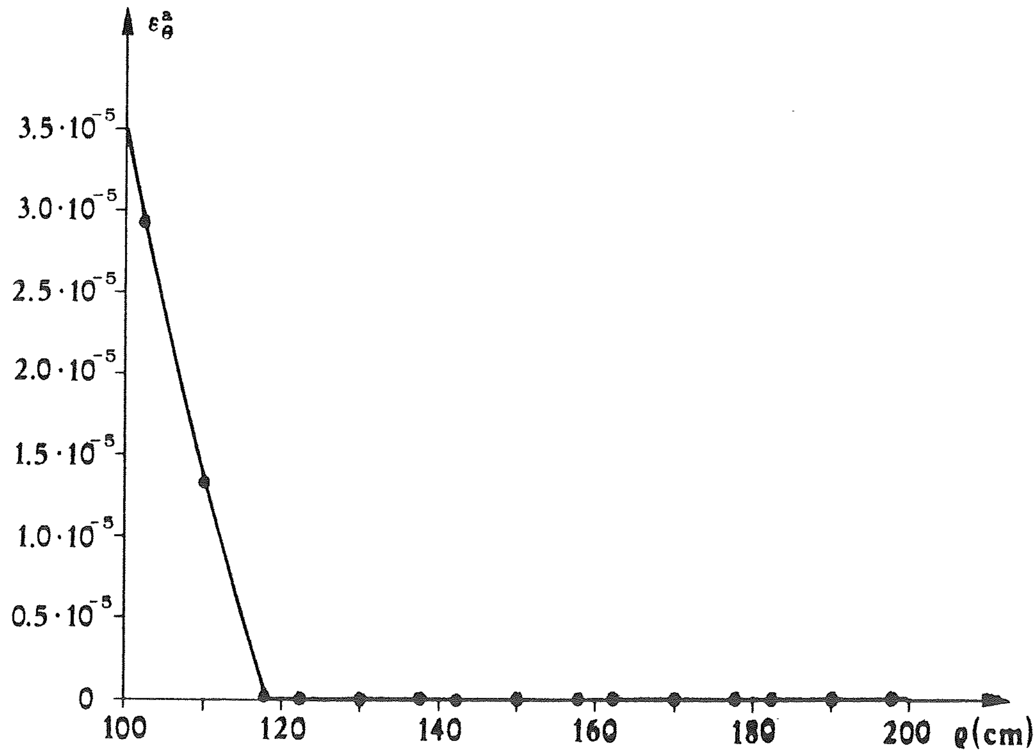

Figure 7. Circumferential inelastic strain vs. ρ.

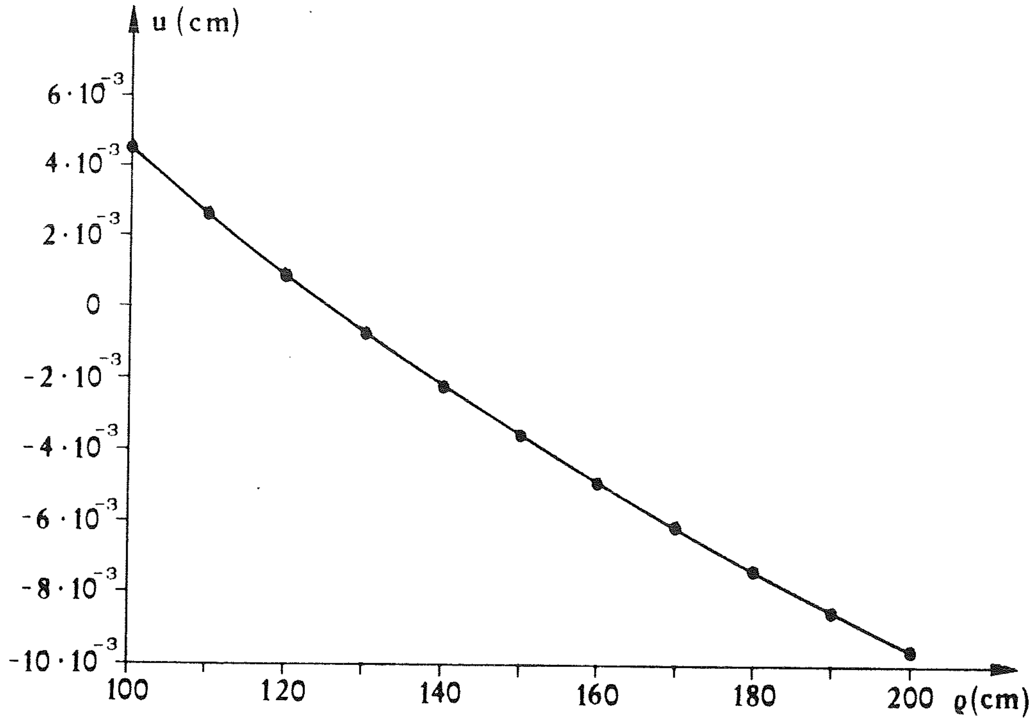

Figure 8. Radial displacement vs. ρ.

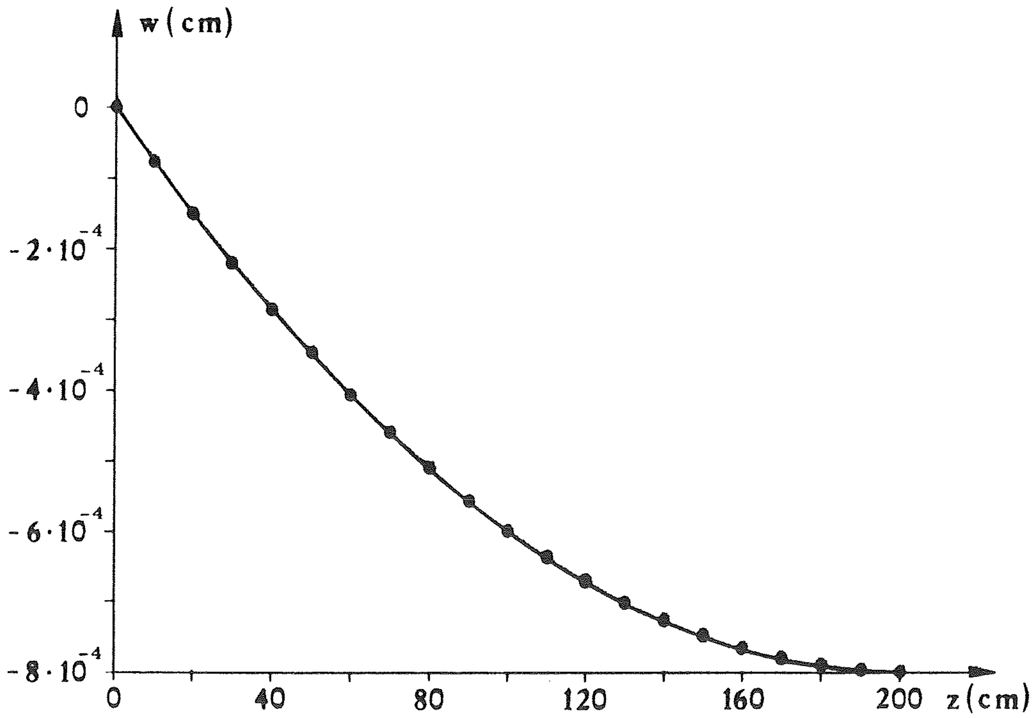

Figure 9. Axial displacement vs. z.

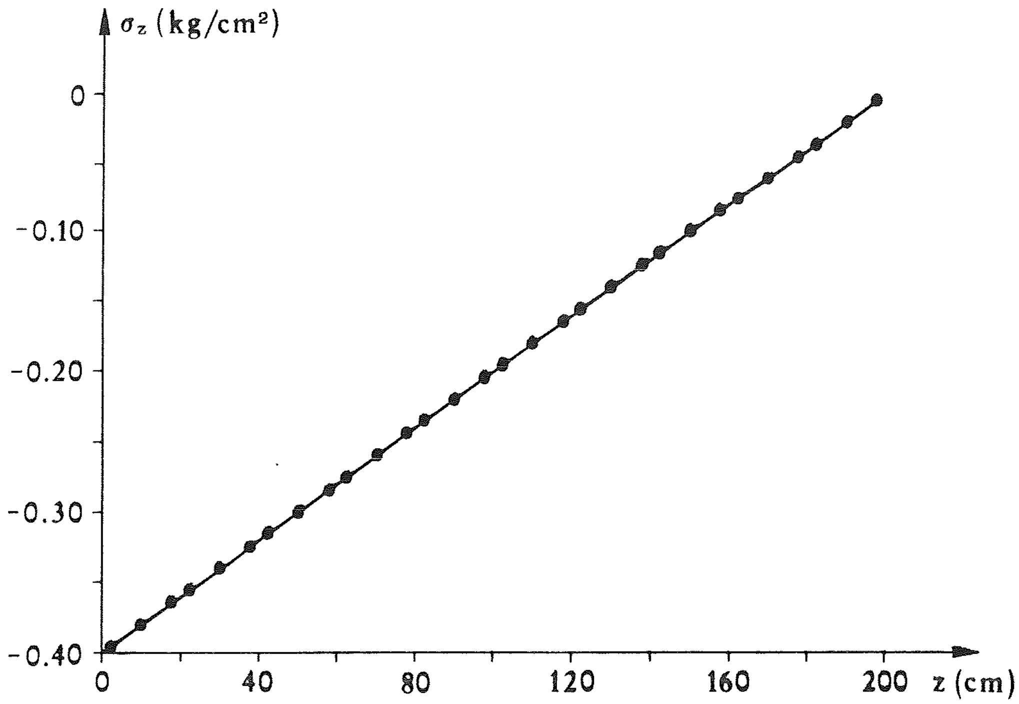

Figure 10. Axial stress vs. z.

Example 3. *Emispherical dome subjected to its own weight and to a point load at the crown.* We consider a hemispherical dome Ω having inner radius a and outer radius b, with specific weight p, subjected to a point load placed at the crown A (Figure 11). Moreover, we suppose that the springings are fixed, and that the material is not resistent to tension, *i. e.* we suppose $\sigma = 0$.

The dome has been discretized into 440 isoparametric axisymmetric elements with four nodes and four Gauss; the mesh is finer near the springings and the keystone. For the numerical calculation of the solution, the following values of the constants have been used:

$a = 400$ cm, $b = 440$ cm,
$p = 0.002$ Kg/cm^3.

We have performed an incremental analysis; at the first load increment only the weight of the dome has been assigned and, successively, incremental point loads, each equal to 50 Kg, have been superimposed at the crown.

Under the action of the weight, inelastic deformations arise along the intrados, near the keystone. Successively, as the point load increases, inelastic deformations arise along the extrados, in proximity to the haunches. Collapse is reached after 218 load increments, corresponding to the load f_s = 10900 Kg. For the determination of the line of thrust, 100 equidistant normal sections have been considered. For each of them, the normal force N and the bending moment M have been calculated, by means of an integration composite trapezoidal open rule, using 50 intervals. Then, the distance e of the line of thrust from the mean line of the dome has been obtained from the well-known relation $e = M/N$. In Figures 12 and 13 the line of thrust is sketched, relative to the 1st and 219th load increment, respectively. In the first case, when the dome is subjected to its own weight alone, the line of thrust is almost entirely contained within the middle-third. In the second case, when the point load reaches the value f_s, the line of thrust is still contained within the dome, but it touches the extrados at the crown and the springings, and the intrados at the haunches ($\theta \approx 48.8°$ and $\theta \approx 131.2°$). Figure 14 shows the behaviour of the distance e of the line of thrust from the mean line as a function of the point load f for $\theta \approx 48.8°$ and $\theta \approx 131.2°$.

The dome behaves (Figure 15) as a serie off "orange-slice"arches [10] and from Figure 14 it can be seen the formation of a five-hinge mechanism. The positions of the hinges being so determined, allow for precise determination of an upper bound value of the collapse load. This upper bound f_c may be easily determined considering each slice arch of arbitrary thickness $\Delta\varphi$ as constituted by four rigid pieces internally and externally connected by hinges and setting the virtual work of the external loads at the instant of collapse equal to zero [11]. Using this procedure, we obtain $f_c = 11508.$ Kg.

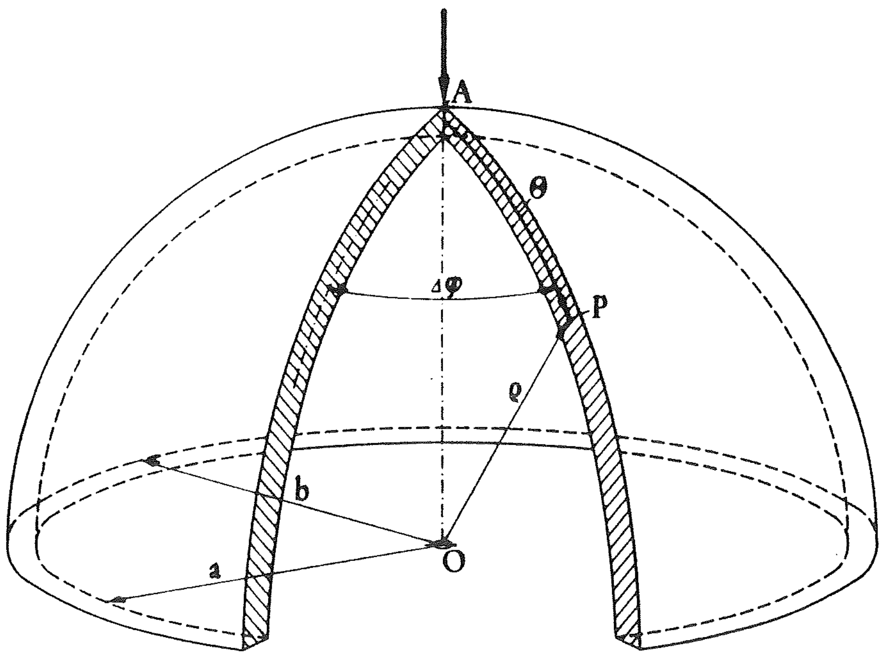

Figure 11. Hemispherical dome subjected to its own weight and to a point load.

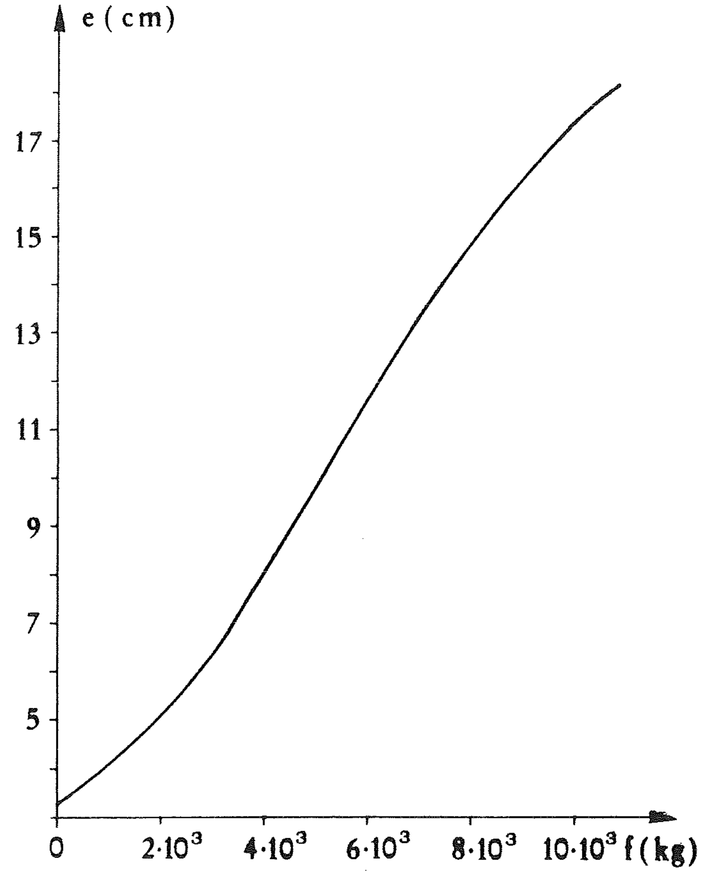

Figure 14. e vs. the point load f, for θ ≈ 48.8° and θ ≈ 131.2°.

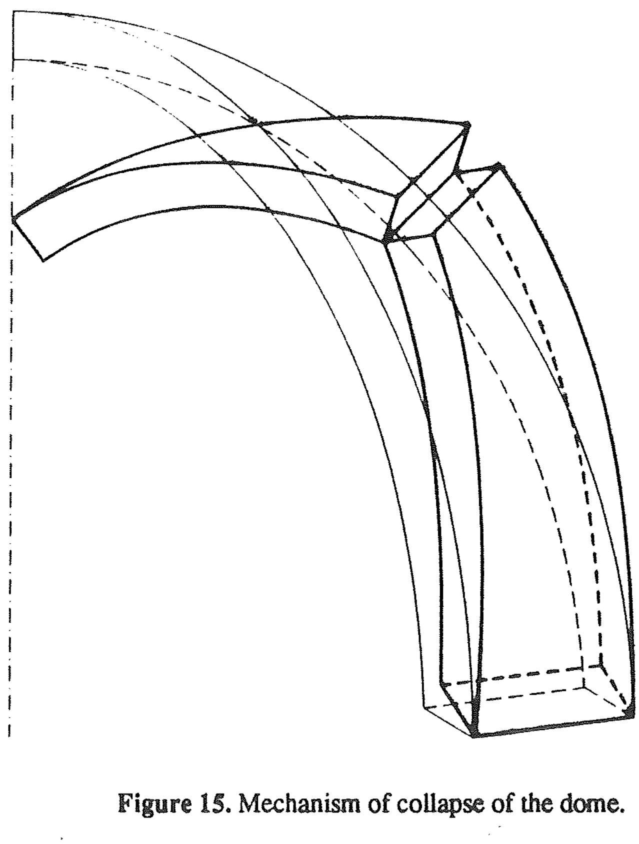

Figure 15. Mechanism of collapse of the dome.

Appendix

Starting from the expression of the derivative of the stress calculated in (3.10), (3.11) and (3.14), we are able to calculate the symmetric matrix D of the engineering components of the fourth-order tensor $\mathbb{D} = D_E T$ for three-dimensional problems; its components are

$$D_{11} = \mathbb{D}_{1111}, \quad D_{12} = \mathbb{D}_{1122}, \quad D_{13} = \mathbb{D}_{1133},$$

$$D_{14} = \frac{\mathbb{D}_{1112} + \mathbb{D}_{1121}}{2}, \quad D_{15} = \frac{\mathbb{D}_{1113} + \mathbb{D}_{1131}}{2}, \quad D_{16} = \frac{\mathbb{D}_{1123} + \mathbb{D}_{1132}}{2}$$

$$D_{22} = \mathbb{D}_{2222}, \quad D_{23} = \mathbb{D}_{2233}, \quad D_{24} = \frac{\mathbb{D}_{2212} + \mathbb{D}_{2221}}{2}$$

$$\text{(A.1)} \quad D_{25} = \frac{\mathbb{D}_{2213} + \mathbb{D}_{2231}}{2}, \quad D_{26} = \frac{\mathbb{D}_{2223} + \mathbb{D}_{2232}}{2}, \quad D_{33} = \mathbb{D}_{3333},$$

$$D_{34} = \frac{\mathbb{D}_{3312} + \mathbb{D}_{3321}}{2}, \quad D_{35} = \frac{\mathbb{D}_{3313} + \mathbb{D}_{3331}}{2}, \quad D_{36} = \frac{\mathbb{D}_{3323} + \mathbb{D}_{3332}}{2},$$

$$D_{44} = \frac{\mathbb{D}_{1212} + \mathbb{D}_{1221}}{2}, \quad D_{45} = \frac{\mathbb{D}_{1213} + \mathbb{D}_{1231}}{2}, \quad D_{46} = \frac{\mathbb{D}_{1223} + \mathbb{D}_{1232}}{2},$$

$$D_{55} = \frac{\mathbb{D}_{1313} + \mathbb{D}_{1331}}{2}, \quad D_{56} = \frac{\mathbb{D}_{1323} + \mathbb{D}_{1332}}{2}, \quad D_{66} = \frac{\mathbb{D}_{2323} + \mathbb{D}_{2332}}{2}.$$

The components of $\mathbb{D}$ are

$$\text{(A.2)} \quad \mathbb{D}_{ijkl} = \alpha_1 (\mathbf{I} \otimes \mathbf{I})_{ijkl} + \alpha_2 (\mathbf{I} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{I})_{ijkl} + \alpha_3 (\mathbf{E} \otimes \mathbf{E})_{ijkl} +$$

$$\alpha_4 (\mathbf{I} \otimes \mathbf{E}^2 + \mathbf{E}^2 \otimes \mathbf{I})_{ijkl} + \alpha_5 (\mathbf{E} \otimes \mathbf{E}^2 + \mathbf{E}^2 \otimes \mathbf{E})_{ijkl} + \alpha_6 (\mathbf{E}^2 \otimes \mathbf{E}^2)_{ijkl} +$$

$$\alpha_7 \mathbb{1}_{ijkl} + \alpha_8 \mathbb{E}_{ijkl}$$

where

if $\mathbf{E} \in \mathcal{R}_1$, then $\quad \alpha_1 = \lambda, \ \alpha_i = 0, i \neq 1, 7, \ \alpha_7 = 2\mu;$

if $\mathbf{E} \in \mathcal{R}_2$, then $\quad \alpha_i = 0, i = 1, 8;$

$$\alpha_1 = \frac{\partial \beta_0}{\partial I_1} = \frac{1}{\beta_{0d}^2} \left(\frac{\partial \beta_{0n}}{\partial I_1} \beta_{0d} - \beta_{0n} \frac{\partial \beta_{0d}}{\partial I_1} \right),$$

$$\alpha_2 = 2\frac{\partial \beta_0}{\partial I_2} = \frac{2}{\beta_{0d}^2}\left(\frac{\partial \beta_{0n}}{\partial I_2}\beta_{0d} - \beta_{0n}\frac{\partial \beta_{0d}}{\partial I_2}\right),$$

$$\alpha_3 = 2\frac{\partial \beta_1}{\partial I_2} = \frac{2}{\beta_{1d}^2}\left(\frac{\partial \beta_{1n}}{\partial I_2}\beta_{1d} - \beta_{1n}\frac{\partial \beta_{1d}}{\partial I_2}\right),$$

if $E \in \mathcal{R}_3$ or $E \in \mathcal{R}_4$, then

$$\alpha_4 = 3\frac{\partial \beta_0}{\partial I_3} = \frac{3}{\beta_{0d}^2}\left(\frac{\partial \beta_{0n}}{\partial I_3}\beta_{0d} - \beta_{0n}\frac{\partial \beta_{0d}}{\partial I_3}\right),$$

$$\alpha_5 = 3\frac{\partial \beta_1}{\partial I_3} = \frac{3}{\beta_{1d}^2}\left(\frac{\partial \beta_{1n}}{\partial I_3}\beta_{1d} - \beta_{1n}\frac{\partial \beta_{1d}}{\partial I_3}\right),$$

$$\alpha_6 = 3\frac{\partial \beta_2}{\partial I_3} = \frac{3}{\beta_{2d}^2}\left(\frac{\partial \beta_{2n}}{\partial I_3}\beta_{2d} - \beta_{2n}\frac{\partial \beta_{2d}}{\partial I_3}\right),$$

$$\alpha_7 = \beta_1 , \qquad \alpha_8 = \beta_2 .$$

In the previous expressions quantities β_{in} and β_{id}, for i = 0, 1, 2, are numerators and denominators of β_i, respectively:

$$\beta_{0n} = \kappa \{2I_3 + I_1^3 - 3I_1 I_2 + \varepsilon \frac{(1+\alpha)}{2+3\alpha} [\frac{8}{3}\chi^2 \cos^2\omega + \frac{2}{3\sqrt{3}}\chi I_1 \cos\omega - \frac{I_1^2}{9} +$$

$$\frac{\alpha}{1+\alpha}(\frac{I_1^2}{9} + \frac{2}{3\sqrt{3}}\chi I_1 \cos\omega + \frac{1}{3}\chi^2(4\cos^2\omega - 3))]\},$$

$$\beta_{0d} = \chi^2 (4\cos^2\omega - 1),$$

$$\beta_{1n} = \kappa \{(\frac{2}{3}I_1 + \frac{2}{\sqrt{3}}\chi\cos\omega)(\frac{\varepsilon}{2+3\alpha} - \frac{I_1}{3} + \frac{2}{\sqrt{3}}\chi\cos\omega)\},$$

$$\beta_{1d} = \chi^2 (4\cos^2\omega - 1),$$

$$\beta_{2n} = \kappa \{\frac{I_1}{3} - \frac{2}{\sqrt{3}}\chi\cos\omega - \frac{\varepsilon}{2+3\alpha}\},$$

$$\beta_{2d} = \chi^2 (4\cos^2\omega - 1)$$

if $E \in \mathcal{R}_3$; and

$$\beta_{0n} = \xi \{-2(2+3\alpha)(2I_3 + I_1^3 - 3I_1 I_2) + 2\alpha\chi[\frac{\sqrt{3}}{9}I_1^2(\cos\omega +$$

$$\sqrt{3}\sin\omega) + \frac{I_1\chi}{3}(2\sqrt{3}\cos\omega\sin\omega - 2\cos^2\omega + 1) -$$

$$\frac{2}{\sqrt{3}}\chi^2(\cos\omega + \sqrt{3}\sin\omega)] - \varepsilon[-\frac{2}{9}I_1^2 - \frac{4}{3}\chi^2\cos\omega(\sqrt{3}\sin\omega - \cos\omega) +$$

$$\frac{2}{3\sqrt{3}}I_1\chi(\cos\omega + \sqrt{3}\sin\omega)] - 2\alpha\chi^2\sin\omega(\sqrt{3}\cos\omega + \sin\omega)]\},$$

$$\beta_{0d} = 4\chi^2\sin\omega(\sin\omega + \sqrt{3}\cos\omega),$$

$$\beta_{1n} = 2\xi\{\alpha I_2 + \frac{4\chi I_1}{3\sqrt{3}}(\cos\omega + \sqrt{3}\sin\omega) +$$

$$\frac{2}{9}I_1^2 + \frac{2}{3}\chi^2(3 - 2\cos^2\omega + 2\sqrt{3}\cos\omega\sin\omega) + (2 + 3\alpha)[\frac{1}{9}I_1^2 -$$

$$-\frac{I_1\chi}{3\sqrt{3}}(\cos\omega + \sqrt{3}\sin\omega) + \frac{2}{3}\chi^2\cos\omega(\sqrt{3}\sin\omega - \cos\omega)] -$$

$$\varepsilon[\frac{2}{3}I_1 - \frac{\chi}{\sqrt{3}}(\cos\omega + \sqrt{3}\sin\omega)]\},$$

$$\beta_{1d} = 4\chi^2\sin\omega(\sin\omega + \sqrt{3}\cos\omega),$$

$$\beta_{2n} = -2\xi\{\frac{2 + 3\alpha}{3}I_1 + \frac{2\chi}{\sqrt{3}}(\cos\omega + \sqrt{3}\sin\omega) - \varepsilon\},$$

$$\beta_{2d} = 4\chi^2\sin\omega(\sin\omega + \sqrt{3}\cos\omega)$$

if $E \in \mathbb{R}_4$.

The derivatives of β_{in} and β_{id} for i = 0, 1, 2, with respect to I_1, I_2 and I_3 are calculated taking into account the expressions of the derivatives of χ and ω with respect to I_1, I_2 and I_3:

$$\frac{\partial\chi}{\partial I_1} = -\frac{I_1}{6\chi}, \qquad \frac{\partial\chi}{\partial I_2} = \frac{1}{4\chi}, \qquad \frac{\partial\chi}{\partial I_3} = 0;$$

$$\frac{\partial\omega}{\partial I_1} = \frac{\sqrt{3}}{2\chi^5}\left(1 - \frac{27\gamma^2}{4\chi^6}\right)^{-1/2}\left\{\left(\frac{2}{9}I_1^2 - \frac{1}{3}I_2\right)\chi^2 + \frac{1}{2}I_1\gamma\right\},$$

$$\frac{\partial\omega}{\partial I_2} = -\frac{\sqrt{3}}{2\chi^5}\left(1 - \frac{27\gamma^2}{4\chi^6}\right)^{-1/2}\left(\frac{1}{3}I_1\chi^2 + \frac{3}{4}\gamma\right),$$

$$\frac{\partial \omega}{\partial I_3} = \frac{\sqrt{3}}{6\chi^3}\left(1 - \frac{27\gamma^2}{4\chi^6}\right)^{-1/2}.$$

The components of matrix D for two-dimensional problems, obtained starting from (4.6) and (4.10), are

$$\text{(A.3)}\quad \begin{aligned} D_{11} &= \alpha_1 + 2\alpha_2 E_{11} + \alpha_3 E_{11}^2 + \alpha_4 , \\ D_{12} &= \alpha_1 + \alpha_2 (E_{11} + E_{22}) + \alpha_3 E_{11} E_{22} , \\ D_{13} &= \alpha_2 E_{12} + \alpha_3 E_{11} E_{12} , \\ D_{22} &= \alpha_1 + 2\alpha_2 E_{22} + \alpha_3 E_{22}^2 + \alpha_4 , \\ D_{23} &= \alpha_2 E_{12} + \alpha_3 E_{22} E_{12} , \\ D_{33} &= \alpha_3 E_{12}^2 + \frac{\alpha_4}{2} , \end{aligned}$$

where E_{ij} are the components of E with respect to the basis $\{\mathbf{g}_1, \mathbf{g}_2\}$ and the scalar functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are

(A.4)$_1$ if $E \in \mathcal{S}_1$, then $\alpha_1 = \lambda, \quad \alpha_2 = \alpha_3 = 0; \quad \alpha_4 = 2\mu;$

(A.4)$_2$ if $E \in \mathcal{S}_2$, then $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0;$

(A.4)$_3$ if $E \in \mathcal{S}_3$, then

$$\alpha_1 = \frac{\varphi}{2}\,\frac{I_1(3I_2 - I_1^2) - \dfrac{\varepsilon}{(1+\alpha)} I_2}{(2I_2 - I_1^2)^{3/2}},$$

$$\alpha_2 = \varphi\,\frac{-I_2 + \dfrac{\varepsilon}{2(1+\alpha)} I_1}{(2I_2 - I_1^2)^{3/2}},$$

$$\alpha_3 = \varphi\,\frac{I_1 - \dfrac{\varepsilon}{1+\alpha}}{(2I_2 - I_1^2)^{3/2}},$$

$$\alpha_4 = \varphi\,\frac{-I_1 + \sqrt{2I_2 - I_1^2} + \dfrac{\varepsilon}{1+\alpha}}{2\sqrt{2I_2 - I_1^2}};$$

for the plane strain, and

(A.5)$_1$ if $E \in \mathcal{T}_1$, then $\alpha_1 = 2\lambda/(2+\alpha)$, $\alpha_2 = \alpha_3 = 0$, $\alpha_4 = 2\mu$;

(A.5)$_2$ if $E \in \mathcal{T}_2$, then $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$;

(A.5)$_3$ if $E \in \mathcal{T}_3$, then

$$\alpha_1 = \frac{\varphi_1}{2} \frac{I_1(3I_2 - I_1^2) - \frac{\varepsilon(2+\alpha)}{2(2+3\alpha)} I_2)}{(2I_2 - I_1^2)^{3/2}},$$

$$\alpha_2 = \varphi_1 \frac{-I_2 + \frac{\varepsilon(2+\alpha)}{2(2+3\alpha)} I_1}{(2I_2 - I_1^2)^{3/2}},$$

$$\alpha_3 = \varphi_1 \frac{I_1 - \frac{\varepsilon(2+\alpha)}{2+3\alpha}}{(2I_2 - I_1^2)^{3/2}},$$

$$\alpha_4 = \varphi_1 \frac{-I_1 + \sqrt{2I_2 - I_1^2} + \frac{\varepsilon(2+\alpha)}{2+3\alpha}}{2\sqrt{2I_2 - I_1^2}};$$

for the plane stress.

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