

ISTITUTO PER LA SCIENZA E TECNOLOGIA DEI PLASMI

Title

General form of the alternative expectation formula when
the variables are defined on the whole real axis

Author

De Vecchi Luca

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Istituto per la Scienza e Tecnologia dei Plasmi

Sede di Milano
Via Cozzi, 53 – 20125 Milano

Sede Secondaria Padova
Corso Stati Uniti, 4 - 35127 Padova

Sede Secondaria Bari
Via Amendola, 122/D - 70126 Bari

- **Abstract**

Hong (2015) showed that the expectation of the product of two nonnegative random variables can be expressed as a double integral involving their joint survival function. In this paper, I extend this result to random variables defined on the entire real line. The obtained identity decomposes the expectation into four terms corresponding to the four quadrants of the Cartesian plane and combines probabilities associated with both joint distribution and joint survival functions in a symmetric way. Three different proofs are presented, all based on a decomposition of the integration domain into four quadrants. The first is based on indicator functions. The second follows the first approach introduced by Hong (2015) for nonnegative random variables and extends it to the whole real axis. The third, extending the second approach of Hong (2015), relies on integration by parts and provides an alternative derivation of the same result. The proposed identity generalizes the alternative expectation formula for a large class real-valued random variables. Extensions to multivariate settings and higher-order moments are discussed, and a numerical verification based on the bivariate normal distribution is reported.

(abstract generated with the support of AI Copilot)

- **Introduction**

The alternative expectation formula for non-negative random variables has been discussed by Hong (2012) [1], Hong (2015) [2] in “The American Statistician”, and earlier references identified by Nadarajah (2016) [3], always in “The American Statistician”. In this paper I derive the corresponding bivariate identities when the two variables are defined on the whole real axis, that is the general form of the alternative expectation formula for bivariate case, with extensions to multivariate case and higher-order moments. I present three possible proofs below, of which the last two are extensions of Hong (2015) [2], after a short recap valid for univariate distributions.

- **Recap for univariate case**

As reported in many statistic books – B.V. Gnedenko (1969)[4] - we have for univariate case

$$E[X] = \int_{-\infty}^{+\infty} xf(x)dx = - \int_{-\infty}^0 F(x)dx + \int_0^{+\infty} (1 - F(x))dx$$

Where

- X is a random variable and $E[X]$ his expectation (in the following we usually use lowercase letters except when we need to specify that the variable X is greater than (or less than) a given value x)
- $f(x)$ is the probability density function of the variable x ;
- $F(x)$ is the cumulative function of the variable x , defined as

$$F(x) = P(X \leq x) = 1 - P(X > x) = 1 - S(x) = \int_{-\infty}^x f(t)dt$$

- $S(x) = P(X > x)$ is the survival function of the random variable X

In fact

$$E[x] = \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^0 xf(x)dx + \int_0^{+\infty} xf(x)dx = \int_{-\infty}^0 x dF(x) - \int_0^{+\infty} x d(1 - F(x))$$

Integrating by parts the two integrals on the right side we obtain

$$\int_{-\infty}^0 x dF(x)$$

$$m = x$$

$$dn = dF(x)$$

$$dm = dx$$

$$n = F(x)$$

$$\int_{-\infty}^0 x dF(x) = [xF(x)]_{-\infty}^0 - \int_{-\infty}^0 F(x)dx = - \int_{-\infty}^0 F(x)dx$$

$$\int_0^{+\infty} x d(1 - F(x))$$

$$m = x$$

$$dn = d(1 - F(x))$$

$$dm = dx$$

$$n = 1 - F(x)$$

$$\int_0^{+\infty} x d(1 - F(x)) = [x(1 - F(x))]_0^{+\infty} - \int_0^{+\infty} (1 - F(x)) dx = - \int_0^{+\infty} (1 - F(x)) dx$$

So

$$\begin{aligned} E[x] &= \int_{-\infty}^{+\infty} x f(x) dx = \int_{-\infty}^0 x f(x) dx + \int_0^{+\infty} x f(x) dx \\ &= \int_{-\infty}^0 x dF(x) - \int_0^{+\infty} x d(1 - F(x)) = - \int_{-\infty}^0 F(x) dx + \int_0^{+\infty} (1 - F(x)) dx \end{aligned}$$

and finally we have for univariate case the following identity, that it's valid on the whole real axis for a large class of distributions – Hong (2012) [1].

$$E[x] = - \int_{-\infty}^0 F(x) dx + \int_0^{+\infty} (1 - F(x)) dx$$

- **The bivariate case**

For bivariate case if the two random variables are non-negative, Hong (2015) [2] proofs that

$$E[xy] = \int_0^{+\infty} \int_0^{+\infty} xy f(x, y) dy dx = \int_0^{+\infty} \int_0^{+\infty} P(X > u, Y > v) dv du = \int_0^{+\infty} \int_0^{+\infty} S(u, v) dv du$$

where $S(u, v)$ is the survival function of $f(x, y)$.

Always for bivariate case, but if the two random variables are defined on the whole real axis, I extend the previous identity and proof in the following that:

$\begin{aligned} E[xy] &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f(x, y) dy dx \\ &= \int_0^{+\infty} du \int_0^{+\infty} P(X > u, Y > v) dv - \int_0^{+\infty} du \int_{-\infty}^0 P(X > u, Y \leq v) dv \\ &\quad - \int_{-\infty}^0 du \int_0^{+\infty} P(X \leq u, Y > v) dv + \int_{-\infty}^0 du \int_{-\infty}^0 P(X \leq u, Y \leq v) dv \\ &= \int_0^{+\infty} \int_0^{+\infty} P(X > u, Y > v) dv du - \int_0^{+\infty} \int_{-\infty}^0 P(X > u, Y \leq v) dv du \\ &\quad - \int_{-\infty}^0 \int_0^{+\infty} P(X \leq u, Y > v) dv du + \int_{-\infty}^0 \int_{-\infty}^0 P(X \leq u, Y \leq v) dv du \end{aligned}$	(1)
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Note the symmetry of equation (1): to each relation of "less than or equal" corresponds an integral between minus infinity and zero while to each relation of "greater than" corresponds an integral between zero and plus infinity. This is analogous to the univariate case, as if some parts of the expression could be attributed to the cumulative distribution function and other parts to the survival function.

Equation (1) generalizes the identity of Hong (2015) [2] from nonnegative random variables to real random variables defined on the whole real axis. To the best of my knowledge, equation (1) is an original contribution and provides, as already said, an extension of Hong (2015) to random variables defined on the entire real line.

1. **First Proof of equation (1)**

With obvious meaning of symbols, we have

$ \begin{aligned} E[xy] &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf(x, y)dydx \\ &= \int_0^{+\infty} \int_0^{+\infty} xyf(x, y)dydx + \int_0^{+\infty} \int_{-\infty}^0 xyf(x, y)dydx \\ &\quad + \int_{-\infty}^0 \int_0^{+\infty} xyf(x, y)dydx + \int_{-\infty}^0 \int_{-\infty}^0 xyf(x, y)dydx \end{aligned} $	(2)
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We've simply decomposed the double integral of the left side in four parts that correspond to the four quadrants of the Cartesian plane.

Observing now that:

- if x and y > 0, we have the following identities

$$\begin{aligned}
 x &= \int_0^{+\infty} 1_{x>u} du = \int_0^x 1 * du + \int_x^{+\infty} 0 * du \\
 y &= \int_0^{+\infty} 1_{y>v} dv = \int_0^y 1 * dv + \int_y^{+\infty} 0 * dv
 \end{aligned}$$

- while, if x and y are < 0, we obtain

$$x = - \int_{-\infty}^0 1_{x<u} du = - \int_{-\infty}^x 0 * du - \int_x^0 1 * du$$

$$y = - \int_{-\infty}^0 1_{y < v} dv = - \int_{-\infty}^y 0 * dv - \int_y^0 1 * dv$$

Substituting the four above identities in (2) we have

$$\begin{aligned} \int_0^{+\infty} \int_0^{+\infty} xyf(x, y) dy dx \\ &= \int_0^{+\infty} x \left(\int_0^{+\infty} yf(x, y) dy \right) dx \\ &= \int_0^{+\infty} \left(\int_0^{+\infty} 1_{x > u} du \right) \left(\int_0^{+\infty} \left(\int_0^{+\infty} 1_{y > v} dv \right) f(x, y) dy \right) dx \\ &= \int_0^{+\infty} \left(\int_0^{+\infty} 1_{x > u} \left(\int_0^{+\infty} f(x, y) \left(\int_0^{+\infty} 1_{y > v} dv \right) dy \right) du \right) dx \end{aligned}$$

Hence, as

- the order of integration is v, y, u and finally x
- in multiple integrals the symbols of differential have to be read from right to left,

the expression is equivalent to

$$\begin{aligned} \int_0^{+\infty} \int_0^{+\infty} xyf(x, y) dy dx \\ &= \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} 1_{x > u} 1_{y > v} f(x, y) dv dy du dx \\ &= \int_0^{+\infty} \int_0^{+\infty} \left(\int_u^{+\infty} \int_v^{+\infty} f(x, y) dy dx \right) dv du = \int_0^{+\infty} \int_0^{+\infty} P(X > u, Y > v) du dv \end{aligned}$$

That is the result of Hong (2015) [2]

Furthermore

$$\begin{aligned} \int_0^{+\infty} \int_{-\infty}^0 xyf(x, y) dy dx &= \int_0^{+\infty} x \left(\int_{-\infty}^0 yf(x, y) dy \right) dx \\ &= \int_0^{+\infty} \left(\int_0^{+\infty} 1_{x > u} du \right) \left(\int_{-\infty}^0 \left(- \int_{-\infty}^0 1_{y < v} dv \right) f(x, y) dy \right) dx \end{aligned}$$

with the above order of integration, so that

$$\begin{aligned} \int_0^{+\infty} \int_{-\infty}^0 xyf(x, y) dy dx &= - \int_0^{+\infty} \int_0^{+\infty} \int_{-\infty}^0 \int_{-\infty}^0 1_{x > u} 1_{y < v} f(x, y) dv dy du dx = \\ &= - \int_0^{+\infty} \int_{-\infty}^0 \left(\int_u^{+\infty} \int_{-\infty}^v f(x, y) dy dx \right) dv du = - \int_0^{+\infty} \int_{-\infty}^0 P(X > u, Y \leq v) dv du \end{aligned}$$

and, in the same way,

$$\begin{aligned}
\int_{-\infty}^0 \int_0^{+\infty} xyf(x, y) dy dx &= \int_{-\infty}^0 x \left(\int_0^{+\infty} yf(x, y) dy \right) dx \\
&= \int_{-\infty}^0 \left(- \int_{-\infty}^0 1_{x < u} du \right) \left(\int_0^{+\infty} \left(\int_0^{+\infty} 1_{y > v} dv \right) f(x, y) dy \right) dx \\
\int_{-\infty}^0 \int_0^{+\infty} xyf(x, y) dy dx &= - \int_{-\infty}^0 \int_{-\infty}^0 \int_0^{+\infty} \int_0^{+\infty} 1_{x < u} 1_{y > v} f(x, y) dv dy du dx \\
&= - \int_{-\infty}^0 \int_0^{+\infty} \left(\int_{-\infty}^u \int_v^{+\infty} f(x, y) dy dx \right) dv du = - \int_{-\infty}^0 \int_0^{+\infty} P(X \leq u, Y > v) dv du
\end{aligned}$$

and again

$$\begin{aligned}
\int_{-\infty}^0 \int_{-\infty}^0 xyf(x, y) dy dx &= \int_{-\infty}^0 x \left(\int_{-\infty}^0 yf(x, y) dy \right) dx \\
&= \int_{-\infty}^0 \left(- \int_{-\infty}^0 1_{x < u} du \right) \left(\int_{-\infty}^0 \left(- \int_{-\infty}^0 1_{y < v} dv \right) f(x, y) dy \right) dx \\
\int_{-\infty}^0 \int_{-\infty}^0 xyf(x, y) dy dx &= \int_{-\infty}^0 \int_{-\infty}^0 \int_{-\infty}^0 \int_{-\infty}^0 1_{x < u} 1_{y < v} f(x, y) dx dy du dv \\
&= \int_{-\infty}^0 \int_{-\infty}^0 \left(\int_{-\infty}^u \int_{-\infty}^v f(x, y) dy dx \right) dv du = \int_{-\infty}^0 \int_{-\infty}^0 P(X \leq u, Y \leq v) dv du
\end{aligned}$$

Hence the proof of (1) is completed.

2. **Second Proof of equation (1)**

Note that we could proceed with the same trick of Hong (2015) [2] for the four integrals of equation (2) as reported in the following.

$$\begin{aligned}
\int_0^{+\infty} \int_0^{+\infty} xyf(x, y) dy dx &= \int_0^{+\infty} \int_0^{+\infty} \left(\int_0^y ds \int_0^x dt f(x, y) \right) dy dx = \int_0^{+\infty} \int_0^x \left(\int_0^{+\infty} \int_0^y f(x, y) ds dy \right) dt dx \\
&= \int_0^{+\infty} \int_t^{+\infty} \left(\int_0^{+\infty} \int_s^{+\infty} f(x, y) dy ds \right) dx dt \\
&= \int_0^{+\infty} \int_0^{+\infty} \left(\int_t^{+\infty} \int_s^{+\infty} f(x, y) dy dx \right) ds dt = \int_0^{+\infty} \int_0^{+\infty} P(T > t, S > s) ds dt
\end{aligned}$$

Where the fundamental analytical step occurs after the third equality sign when the order of integration is changed for the two pairs of double integrals while keeping the integration region unchanged. This is exactly the same proof of Hong (2015) [2] where x and y are swapped

With the other three double integrals of equation (2) we can proceed in the same manner with minus changes as in the following

$$\begin{aligned}
& \int_0^{+\infty} \int_{-\infty}^0 xyf(x,y)dydx \\
&= \int_0^{+\infty} \int_{-\infty}^0 \left(\int_0^y ds \int_0^x dt f(x,y) \right) dydx \\
&= \int_0^{+\infty} \int_0^x \left(\int_{-\infty}^0 \int_0^y f(x,y) ds dy \right) dt dx \\
&= - \int_0^{+\infty} \int_0^x \left(\int_{-\infty}^0 \int_y^0 f(x,y) ds dy \right) dt dx \\
&= - \int_0^{+\infty} \int_t^{+\infty} \left(\int_{-\infty}^0 \int_{-\infty}^s f(x,y) dy ds \right) dx dt \\
&= - \int_0^{+\infty} \int_{-\infty}^0 \left(\int_t^{+\infty} \int_{-\infty}^s f(x,y) dy dx \right) ds dt = - \int_0^{+\infty} \int_{-\infty}^0 P(T > t, S \leq s) ds dt
\end{aligned}$$

Where, as above, the fundamental analytical step occurs when the order of integration is changed for the two pairs of double integrals while keeping the integration region unchanged (here after the fourth equality sign)

In the same way

$$\begin{aligned}
& \int_{-\infty}^0 \int_0^{+\infty} xyf(x,y)dydx = \int_{-\infty}^0 \int_0^{+\infty} \left(\int_0^y ds \int_0^x dt f(x,y) \right) dydx \\
&= \int_{-\infty}^0 \int_0^x \left(\int_0^{+\infty} \int_0^y f(x,y) ds dy \right) dt dx \\
&= - \int_{-\infty}^0 \int_x^0 \left(\int_0^{+\infty} \int_0^y f(x,y) ds dy \right) dt dx \\
&= - \int_{-\infty}^0 \int_{-\infty}^t \left(\int_0^{+\infty} \int_s^{+\infty} f(x,y) dy ds \right) dx dt \\
&= - \int_{-\infty}^0 \int_0^{+\infty} \left(\int_{-\infty}^t \int_s^{+\infty} f(x,y) dy dx \right) ds dt = - \int_{-\infty}^0 \int_0^{+\infty} P(T \leq t, S > s) ds dt
\end{aligned}$$

At the end

$$\begin{aligned}
\int_{-\infty}^0 \int_{-\infty}^0 xyf(x,y)dydx &= \int_{-\infty}^0 \int_{-\infty}^0 \left(\int_0^y ds \int_0^x dt f(x,y) \right) dydx \\
&= \int_{-\infty}^0 \int_0^x \left(\int_{-\infty}^0 \int_0^y f(x,y) dsdy \right) dt dx \\
&= \int_{-\infty}^0 \int_x^0 \left(\int_{-\infty}^0 \int_y^0 f(x,y) dsdy \right) dt dx = \int_{-\infty}^0 \int_{-\infty}^t \left(\int_{-\infty}^0 \int_{-\infty}^s f(x,y) dyds \right) dx dt \\
&= \int_{-\infty}^0 \int_{-\infty}^0 \left(\int_{-\infty}^t \int_{-\infty}^s f(x,y) dydx \right) ds dt = \int_{-\infty}^0 \int_{-\infty}^0 P(T \leq t, S \leq s) ds dt
\end{aligned}$$

Hence the proof of (1) is completed

3. Third Proof of equation (1)

We first observe that

$$\frac{\partial^2}{\partial x \partial y} P(X > x, Y > y) = \frac{\partial^2}{\partial x \partial y} S(x, y) = f(x, y)$$

$$\frac{\partial^2}{\partial x \partial y} P(X \leq x, Y \leq y) = \frac{\partial^2}{\partial x \partial y} F(x, y) = f(x, y)$$

$$\frac{\partial^2}{\partial x \partial y} P(X > x, Y \leq y) = -f(x, y)$$

$$\frac{\partial^2}{\partial x \partial y} P(X \leq x, Y > y) = -f(x, y)$$

In fact as

$$P(X > x, Y > y) = \int_x^{+\infty} \int_y^{+\infty} f(u, v) dv du$$

$$P(X \leq x, Y \leq y) = \int_{-\infty}^x \int_{-\infty}^y f(u, v) dv du$$

$$P(X > x, Y \leq y) = \int_x^{+\infty} \int_{-\infty}^y f(u, v) dv du$$

$$P(X \leq x, Y > y) = \int_{-\infty}^x \int_y^{+\infty} f(u, v) dv du$$

deriving twice under the integral sign (first with respect to x, then with respect to y) we obtain the above identities.

But the equation (2) states that

$$\begin{aligned}
E[xy] &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf(x,y)dydx \\
&= \int_0^{+\infty} \int_0^{+\infty} xyf(x,y)dydx + \int_0^{+\infty} \int_{-\infty}^0 xyf(x,y)dydx \\
&\quad + \int_{-\infty}^0 \int_0^{+\infty} xyf(x,y)dydx + \int_{-\infty}^0 \int_{-\infty}^0 xyf(x,y)dydx
\end{aligned}$$

So by substitution we obtain

$$\begin{aligned}
E[xy] &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf(x,y)dydx \\
&= \int_0^{+\infty} \int_0^{+\infty} xy \frac{\partial^2}{\partial x \partial y} P(X > x, Y > y) dy dx \\
&\quad - \int_0^{+\infty} \int_{-\infty}^0 xy \frac{\partial^2}{\partial x \partial y} P(X > x, Y \leq y) dy dx \\
&\quad - \int_{-\infty}^0 \int_0^{+\infty} xy \frac{\partial^2}{\partial x \partial y} P(X \leq x, Y > y) dy dx \\
&\quad + \int_{-\infty}^0 \int_{-\infty}^0 xy \frac{\partial^2}{\partial x \partial y} P(X \leq x, Y \leq y) dy dx
\end{aligned}$$

Let us now consider separately the four integrals of the right side of the equation

First we have

$$\int_0^{+\infty} \int_0^{+\infty} xy \frac{\partial^2}{\partial x \partial y} P(X > x, Y > y) dy dx$$

that is the same expression of Hong (2015) [2] so that we can argue in the same way and get

$$\int_0^{+\infty} \int_0^{+\infty} xy \frac{\partial^2}{\partial x \partial y} P(X > x, Y > y) dy dx = \int_0^{+\infty} \int_0^{+\infty} P(X > u, Y > v) dv du$$

Then, taking the last integral and integrating by parts we have

$$\begin{aligned}
&\int_{-\infty}^0 \int_{-\infty}^0 xy \frac{\partial^2}{\partial x \partial y} P(X \leq x, Y \leq y) dy dx \\
&= \int_{-\infty}^0 x \left(\int_{-\infty}^0 y dy \left(\frac{\partial}{\partial x} P(X \leq x, Y \leq y) \right) \right) dx \\
&= \int_{-\infty}^0 x \left(\lim_{y \rightarrow -\infty} \left[s \frac{\partial}{\partial x} P(X \leq x, Y \leq s) \right]_y^0 - \int_{-\infty}^0 \frac{\partial}{\partial x} P(X \leq x, Y \leq y) dy \right) dx \\
&= \int_{-\infty}^0 x \left(\lim_{y \rightarrow -\infty} \left[s \frac{\partial}{\partial x} P(X \leq x, Y \leq s) \right]_y^0 \right) dx \\
&\quad - \int_{-\infty}^0 x \left(\int_{-\infty}^0 \frac{\partial}{\partial x} P(X \leq x, Y \leq y) dy \right) dx
\end{aligned}$$

where the subscript y of d_y in the second equality emphasizes that the partial differentiation is taken with respect to y . The first term is zero as

$$\begin{aligned} \text{For } y \rightarrow -\infty: \quad 0 &\geq y \frac{\partial}{\partial x} P(X \leq x, Y \leq y) \\ &= y \int_{-\infty}^y \frac{\partial}{\partial s} \left(\frac{\partial}{\partial x} P(X \leq x, Y \leq s) \right) ds = y * \int_{-\infty}^y f(x, s) ds \geq \int_{-\infty}^y s f(x, s) ds \rightarrow 0 \end{aligned}$$

For the second addendum we have

$$\begin{aligned} - \int_{-\infty}^0 x \left(\int_{-\infty}^0 \frac{\partial}{\partial x} P(X \leq x, Y \leq y) dy \right) dx \\ &= \int_{-\infty}^0 \left(- \int_{-\infty}^0 x d_x P(X \leq x, Y \leq y) \right) dy \\ &= \int_{-\infty}^0 \left(- \lim_{x \rightarrow -\infty} [s * P(X \leq s, Y \leq y)]_x^0 + \int_{-\infty}^0 P(X \leq x, Y \leq y) dx \right) dy \\ &= \int_{-\infty}^0 \int_{-\infty}^0 P(X \leq x, Y \leq y) dx dy \end{aligned}$$

$$\begin{aligned} \text{In fact for } x \rightarrow -\infty: \quad 0 &\geq x * P(X \leq x, Y \leq y) \\ &= x \int_{-\infty}^x \frac{\partial}{\partial t} P(X \leq t, Y \leq y) dt \geq \int_{-\infty}^x t \frac{\partial}{\partial t} P(X \leq t, Y \leq y) dt \rightarrow 0 \end{aligned}$$

Obviously d_x indicates that the partial differentiation is taken with respect to x

Now we consider the two double integrals with the minus sign

$$\begin{aligned} - \int_0^{+\infty} \int_{-\infty}^0 xy \frac{\partial^2}{\partial x \partial y} P(X > x, Y \leq y) dy dx \\ &= - \int_0^{+\infty} x \left(\int_{-\infty}^0 y \frac{\partial^2}{\partial x \partial y} P(X > x, Y \leq y) dy \right) dx \\ &= - \int_0^{+\infty} x \left(\int_{-\infty}^0 y d_y \left(\frac{\partial}{\partial x} P(X > x, Y \leq y) \right) \right) dx \\ &= \int_0^{+\infty} x \left(- \lim_{y \rightarrow -\infty} \left[s \frac{\partial}{\partial x} P(X > x, Y \leq s) \right]_y^0 + \int_{-\infty}^0 \frac{\partial}{\partial x} P(X > x, Y \leq y) dy \right) dx \end{aligned}$$

where the subscript y of d_y in the third equality emphasizes that the partial differentiation is taken with respect to y . As above the first term is zero because

$$\begin{aligned} \text{For } y \rightarrow -\infty \quad 0 &\leq y \frac{\partial}{\partial x} P(X > x, Y \leq y) = y \int_{-\infty}^y \frac{\partial}{\partial s} \left(\frac{\partial}{\partial x} P(X > x, Y \leq s) \right) ds \\ &= -y \int_{-\infty}^y f(x, s) ds \leq - \int_{-\infty}^y s f(x, s) ds \rightarrow 0 \end{aligned}$$

In fact

$$y \int_{-\infty}^y f(x, s) ds \geq \int_{-\infty}^y s f(x, s) ds$$

$$-y \int_{-\infty}^y f(x, s) ds \leq - \int_{-\infty}^y s f(x, s) ds$$

For the second addendum we have

$$\begin{aligned} \int_0^{+\infty} x \left(\int_{-\infty}^0 \frac{\partial}{\partial x} P(X > x, Y \leq y) dy \right) dx \\ &= \int_{-\infty}^0 \left(\int_0^{+\infty} x d_x (P(X > x, Y \leq y)) \right) dy \\ &= \int_{-\infty}^0 \left(\lim_{x \rightarrow +\infty} [s P(X > s, Y \leq y)]_0^x - \int_0^{+\infty} P(X > x, Y \leq y) dx \right) dy \\ &= - \int_{-\infty}^0 \int_0^{+\infty} P(X > x, Y \leq y) dx dy \end{aligned}$$

$$\begin{aligned} \text{In fact for } x \rightarrow +\infty \quad 0 \leq x P(X > x, Y \leq y) &= -x \int_x^{+\infty} \frac{\partial}{\partial t} P(X > t, Y \leq y) dt \\ &\leq - \int_x^{+\infty} t \frac{\partial}{\partial t} P(X > t, Y \leq y) dt \rightarrow 0 \end{aligned}$$

In fact remembering that $\frac{\partial}{\partial t} P(X > t, Y \leq y) \leq 0$

$$\begin{aligned} x \int_x^{+\infty} \frac{\partial}{\partial t} P(X > t, Y \leq y) dt &\geq \int_x^{+\infty} t \frac{\partial}{\partial t} P(X > t, Y \leq y) dt \\ -x \int_x^{+\infty} \frac{\partial}{\partial t} P(X > t, Y \leq y) dt &\leq - \int_x^{+\infty} t \frac{\partial}{\partial t} P(X > t, Y \leq y) dt \end{aligned}$$

And again for the second integral with minus sign

$$\begin{aligned}
& - \int_{-\infty}^0 \int_0^{+\infty} xy \frac{\partial^2}{\partial x \partial y} P(X \leq x, Y > y) dy dx \\
& = - \int_{-\infty}^0 x \left(\int_0^{+\infty} y \frac{\partial^2}{\partial x \partial y} P(X \leq x, Y > y) dy \right) dx \\
& = - \int_{-\infty}^0 x \left(\int_0^{+\infty} y d_y \left(\frac{\partial}{\partial x} P(X \leq x, Y > y) \right) \right) dx \\
& = \int_{-\infty}^0 x \left(- \lim_{y \rightarrow +\infty} \left[s \frac{\partial}{\partial x} P(X \leq x, Y > s) \right]_0^y + \int_0^{+\infty} \frac{\partial}{\partial x} P(X \leq x, Y > y) dy \right) dx
\end{aligned}$$

where the subscript y of d_y in the third equality emphasizes that the partial differentiation is taken with respect to y . As above the first term is zero because

$$\begin{aligned}
\text{For } y \rightarrow +\infty \quad 0 & \leq y \frac{\partial}{\partial x} P(X \leq x, Y > y) = -y \int_y^{+\infty} \frac{\partial}{\partial s} \left(\frac{\partial}{\partial x} P(X \leq x, Y > s) \right) ds \\
& = y \int_y^{+\infty} f(x, s) ds \leq \int_y^{+\infty} s f(x, s) ds \rightarrow 0
\end{aligned}$$

As above we treat the second addendum

$$\begin{aligned}
\int_{-\infty}^0 x \left(\int_0^{+\infty} \frac{\partial}{\partial x} P(X \leq x, Y > y) dy \right) dx & = \int_0^{+\infty} \left(\int_{-\infty}^0 x d_x P(X \leq x, Y > y) \right) dy \\
& = \int_0^{+\infty} \left(\lim_{x \rightarrow -\infty} [s P(X \leq s, Y > y)]_x^0 - \int_{-\infty}^0 P(X \leq x, Y > y) dx \right) dy \\
& = - \int_0^{+\infty} \int_{-\infty}^0 P(X \leq x, Y > y) dx dy
\end{aligned}$$

$$\begin{aligned}
\text{In fact for } x \rightarrow -\infty \quad 0 & \geq x P(X \leq x, Y > y) = x \int_{-\infty}^x \frac{\partial}{\partial t} P(X \leq t, Y > y) dt \\
& \geq \int_{-\infty}^x t \frac{\partial}{\partial t} P(X \leq t, Y > y) dt \rightarrow 0
\end{aligned}$$

Hence the proof of (1) is completed.

4. Some extensions

There are two possible extensions to (1): multivariate extensions and high-order moment extensions. In fact if one consider the first proof can easily see that is possible to calculate in the same manner

$$E[x_1 * x_2 * \dots * x_n]$$

In particular if $x_1, x_2 \dots x_n \geq 0$ one obtain the expression of Nadarajah (2016) [3] that I report in the following

$$E[x_1 * x_2 * \dots * x_n] = \int_0^{+\infty} \dots \int_0^{+\infty} S(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n$$

where $S(x_1, x_2, \dots, x_n)$ is the joint survival function of $x_1, x_2 \dots x_n$

As further original contribution note that, if $x_1, x_2 \dots x_n \leq 0$,

$$E[x_1 * x_2 * \dots * x_n] = (-1)^n \int_{-\infty}^0 \dots \int_{-\infty}^0 F(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n$$

where $F(x_1, x_2, \dots, x_n)$ is the joint cumulative function of $x_1, x_2 \dots x_n$. The expressions are congruent with these of one-dimensional and bi-dimensional cases.

Furthermore is possible to calculate in the same manner

$$E[x^m * y^n]$$

with $m, n \in N^+$ observing that

if $x \geq y > 0$

$$\begin{aligned} x^m &= \int_0^{+\infty} 1_{x>u} m u^{m-1} du = \int_0^x m u^{m-1} du + \int_x^{+\infty} 0 * du \\ y^n &= \int_0^{+\infty} 1_{y>v} n v^{n-1} dv = \int_0^y n v^{n-1} dv + \int_y^{+\infty} 0 * dv \end{aligned}$$

While, if x e y sono < 0 ,

$$x^m = - \int_{-\infty}^0 1_{x < u} m u^{m-1} du = - \int_x^0 m u^{m-1} * du - \int_{-\infty}^x 0 * du$$

$$y^n = - \int_{-\infty}^0 1_{y < v} n v^{n-1} dv = - \int_y^0 n v^{n-1} * dv - \int_{-\infty}^y 0 * dv$$

Obviously it's also possible to calculate in the same manner (combining multivariate extensions and high order moment extensions)

$$E[x_1^{m_1} * x_2^{m_2} * \dots * x_n^{m_n}]$$

with $m_1, m_2, \dots, m_n \in N^+$

- **A numerical example**

To strengthen the equation (1) with a numerical example, the bivariate normal distribution was considered and the following table, that numerically verify equation (1), was obtained. If you are interested in the code you can ask for a copy of this scientific relation [5]

Parameters of bivariate normal distribution					Numerical value of the left side of the equation (1)	Numerical value of the right side of the equation (1)	Theoretical value
μ_1	μ_2	σ_1	σ_2	ρ			
0	0	1	1	0	0	-5.55112e-17	0
1	1	1	1	0.7	1.7	1.70001	1.7
-1	-1	1	1	0.5	1.5004	1.5	1.5
0	0	1.5	1.5	0.5	1.12482	1.12482	1.250
0	0	1.5	1.5	0.5	1.1249	1.12502	1.250

Table 1 – Verification table of equation (1) for bivariate normal distribution computing integrals numerically calculated with a software written in Igor Pro - Wavemetrics (a double integral for the left side of equation (1) and four quadrupole integrals for the right side of the same expression). The rounding of the penultimate line improves if the integration range is extended to obtain the results of the last line (theoretical value 1.250). The left part of the expression (1) of the first row is obtained with the Gaussian quadrature.

- **Conclusions**

I have provided three proofs of the equation (1) that is the general form of the alternative expectation formula for a couple of bivariate random variables. The second and third proofs are an extension of the first and second approach on Hong (2015) [2].

For the best of my knowledge equation (1) is an original contribution.

Furthermore I suggest some multivariate extensions, including in particular the case of non-positive variables; this case is the complement of Nadarajah (2016) [3] for non-negative variables

Bibliography

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