

Well-posedness, regularity and asymptotic analyses for a fractional phase field system

PIERLUIGI COLLI^(*)

e-mail: pierluigi.colli@unipv.it

GIANNI GILARDI^(*)

e-mail: gianni.gilardi@unipv.it

^(*) Dipartimento di Matematica “F. Casorati”, Università di Pavia
and Research Associate at the IMATI – C.N.R. Pavia
via Ferrata 5, 27100 Pavia, Italy

*Dedicated to our friend Prof. Dr. Jürgen Sprekels
on the occasion of his 70th birthday
with best wishes*

Abstract

This paper is concerned with a non-conserved phase field system of Caginalp type in which the main operators are fractional versions of two fixed linear operators A and B . The operators A and B are supposed to be densely defined, unbounded, self-adjoint, monotone in the Hilbert space $L^2(\Omega)$, for some bounded and smooth domain Ω , and have compact resolvents. Our definition of the fractional powers of operators uses the approach via spectral theory. A nonlinearity of double-well type occurs in the phase equation and either a regular or logarithmic potential, as well as a non-differentiable potential involving an indicator function, is admitted in our approach. We show general well-posedness and regularity results, extending the corresponding results that are known for the non-fractional elliptic operators with zero Neumann conditions or other boundary conditions like Dirichlet or Robin ones. Then, we investigate the longtime behavior of the system, by fully characterizing every element of the ω -limit as a stationary solution. In the final part of the paper we study the asymptotic behavior of the system as the parameter σ appearing in the operator $B^{2\sigma}$ that plays in the phase equation decreasingly tends to zero. We can prove convergence to a phase relaxation problem at the limit, in which an additional term containing the projection of the phase variable on the kernel of B appears.

Key words: Fractional operators, Allen–Cahn equations, phase field system, well-posedness, regularity, asymptotics.

AMS (MOS) Subject Classification: 35K45, 35K90, 35R11, 35B40.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ denote a bounded, connected and smooth set. We deal with the Cauchy problem for the evolutionary system

$$\partial_t \vartheta + \ell(\varphi) \partial_t \varphi + A^{2r} \vartheta = f \quad (1.1)$$

$$\partial_t \varphi + B^{2\sigma} \varphi + F'(\varphi) = \vartheta \ell(\varphi) \quad (1.2)$$

where A^{2r} and $B^{2\sigma}$, with $r > 0$ and $\sigma > 0$, denote fractional powers of the self-adjoint, monotone and unbounded linear operators A and B , respectively, which are densely defined in $H := L^2(\Omega)$ and are supposed to have compact resolvents.

The above system is a generalization of the well-known phase field system, which models a phase transition process taking place in the container Ω . In this case, one typically has $A^{2r} = B^{2\sigma} = -\Delta$ with, e.g., zero Neumann boundary conditions if no flux through the boundary is assumed for both variables. About the meaning of variables in (1.1)–(1.2), let us notice that the first unknown function ϑ represents the *relative temperature* near some critical value ϑ_c , while φ usually denotes the *order parameter*. Moreover, the real function ℓ represents the *latent heat density* and f is a source term. Finally, F' denotes the derivative of a potential F , which may have a double-well shape.

Thus, the coupled equations (1.1)–(1.2) yield a system of phase field type. From the seminal work of Caginalp and coworkers (see, e.g., [13, 14]) it became clear that phase field systems are particularly suited to represent the dynamics of moving interfaces arising in thermally induced phase transitions. Typical and physically significant examples for the potential F are the so-called *classical regular potential*, the *logarithmic double-well potential*, and the *double obstacle potential*, which are given, in this order, by

$$F_{reg}(s) := \frac{1}{4} (s^2 - 1)^2, \quad s \in \mathbb{R}, \quad (1.3)$$

$$F_{log}(s) := ((1+s) \ln(1+s) + (1-s) \ln(1-s)) - c_1 s^2, \quad s \in (-1, 1), \quad (1.4)$$

$$F_{2obs}(s) := -c_2 s^2 \quad \text{if } |s| \leq 1 \quad \text{and} \quad F_{2obs}(s) := +\infty \quad \text{if } |s| > 1. \quad (1.5)$$

Here, the constants c_i in (1.4) and (1.5) satisfy $c_1 > 1$ and $c_2 > 0$, so that f_{log} and f_{2obs} are nonconvex. In cases like (1.5), one has to split F into a nondifferentiable convex part $\widehat{\beta}$ (the indicator function of $[-1, 1]$, in the present example) and a smooth concave perturbation $\widehat{\pi}$ ($\widehat{\pi}(s) = -c_2 s^2$, $s \in \mathbb{R}$, in (1.5)). Accordingly, one has to replace the derivative of the convex part by the subdifferential and interpret (1.2) as a differential inclusion or, equivalently, as a variational inequality involving $\widehat{\beta}$ rather than its subdifferential. Actually, we will mostly do the latter in this paper.

In fact, in the present paper we discuss the solvability of the initial value problem for the system (1.1)–(1.2) in the framework when both r and σ are positive, by first proving a well-posedness result. This is worked out in a suitable framework, first in the case of a constant ℓ for general operators A^r and B^σ , then for a bounded and Lipschitz continuous nonlinearity ℓ , under some restriction on the domains of A^r and B^σ ; indeed, $D(A^r)$ and $D(B^\sigma)$ have to be contained into appropriate Lebesgue spaces. Then, in a second part of the discussion we show some regularity results and we also investigate the longtime behavior of the system, by fully characterizing the ω -limit. In the final part of the paper

we focus our attention on the analysis, which turns out to be rather delicate and not trivial at all, of the asymptotic behavior of the system (1.1)–(1.2) as the coefficient σ playing in (1.2) decreases to 0. We can prove convergence to a phase relaxation problem at the limit, in the special case of a constant ℓ and also for a concave quadratic function $\widehat{\pi}$. However, the full set of our results is precisely described in the next section, in great detail.

Thus, here we are dealing with fractional operators, which are nowadays a challenging subject for mathematicians: in particular, different variants of fractional operators may be considered and tackled. Let us mention some related contribution, starting from the paper [26], which deals with several definitions of the fractional Laplacian, which is a core example of a class of nonlocal pseudodifferential operators appearing in various areas of theoretical and applied mathematics. We quote some contributions by Servadei and Valdinoci: in [32], a comparison is made between the spectrum of two different fractional Laplacian operators, of which the second one fits in our framework; in [33] the regularity of the weak solution to the fractional Laplace equation is discussed; [31] is concerned with the existence of nontrivial solutions for nonlocal semilinear Dirichlet problem; in [34] the authors show a fractional counterpart to the well-known Brezis–Nirenberg result on the existence of nontrivial solutions to elliptic equations with critical nonlinearities. In [1] a construction of harmonic functions on bounded domains is given for the spectral fractional Laplacian operator. In the paper [11], the authors investigate a nonlinear pseudodifferential boundary value problem in a bounded domain with homogeneous Dirichlet boundary conditions. Regularity results and sharp estimates are discussed in [12] for fractional elliptic equations. Fractional Dirichlet and Neumann type boundary problems associated with the fractional Laplacian are investigated in [23], by demonstrating regularity properties with a spectral approach; this analysis is extended to the fractional heat equation in [24]. The contribution [28] deals with obstacle problems for the spectral fractional Laplacian. The authors of [29, 30] prove regularity up to the boundary for a boundary value problem using the Caputo variant of an integral operator with the Riesz kernel. Some nonlocal problems involving the fractional p -Laplacian and nonlinearities at critical growth are examined in [9]. Fractional porous medium type equations are discussed in [6, 7, 8]: [7] deals with existence, uniqueness and asymptotic behavior of the solutions to an integro-differential equation related to porous medium equations in bounded domains; uniform estimates for positive solutions of a porous medium equation are derived in [8], where the spectral fractional Laplacian with zero Dirichlet boundary data is considered; a quantitative study of nonnegative solutions of the same equation is provided in [6], where decay and positivity, Harnack inequalities, interior and boundary regularity, and asymptotic behavior are investigated.

We point out that there are already some contributions addressing nonlocal variants of Allen–Cahn, Cahn–Hilliard and phase field systems. In [3], Akagi, Schimperna and Segatti introduce a fractional variant of the Cahn–Hilliard equation settled in a bounded domain and complemented with homogeneous Dirichlet boundary conditions of solid type; they prove existence and uniqueness of weak solutions and investigate some significant singular limits as the order of either of the fractional Laplacians appearing in the system approaches zero. In this respect, their results can be compared with our results of Section 7: it is worth mentioning that the authors of [3] use fractional operators not defined via the spectral properties and actually different from ours. In the recent paper [4], for fixed orders of the operators, the same authors show the convergence as time goes to infinity of

each solution to a (single) equilibrium. The contribution [2] deals with a fractional Cahn–Hilliard equation by considering a gradient flow in the negative order Sobolev space $H^{-\alpha}$, $\alpha \in [0, 1]$, where the case $\alpha = 1$ corresponds to the classical Cahn–Hilliard equation and the choice $\alpha = 0$ recovers the Allen–Cahn equation; existence and stability estimates are proved. We also mention the articles [18, 19] that are concerned with nonlocal phase field models for phase separations, using a free energy which arises naturally in the analysis of the large scale limit of systems of interacting particles. Another interesting analysis of a nonstandard and nonlocal Cahn–Hilliard system can be found in [16]. A non-local version of the Cahn–Hilliard equation is treated in [20]; the papers [21, 22] investigate a doubly nonlocal Cahn–Hilliard equation with special kernels in the operators, by focusing on the interaction between the two levels of nonlocality in the corresponding terms. The paper [25] studies numerical solutions to the Allen–Cahn equation with a fractional Laplacian: the authors use the second-order Crank–Nicolson scheme to discretize the equation in time and the second-order central difference scheme for discretization in space. A space-time fractional Allen–Cahn phase-field model that describes the transport of the fluid mixture of two immiscible fluid phases is discussed in [27]; the space and time fractional order parameters control the sharpness and the decay behavior of the interface. We also quote the contribution [15], where melting and solidification for metallurgical processes concerned with phase transitions of pure metals are studied; during the solid phase the metals show an evident ductility and these particular phenomena can be well described by a phase field fractional model, whose evolution has to satisfy a Ginzburg–Landau equation.

In our approach, which follows closely the setting recently used in [17], we work with fractional operators defined via spectral theory. By this, we can easily consider powers of a second-order elliptic operator with either Dirichlet or Neumann or Robin boundary conditions, as well as other operators, e.g., fourth-order ones or systems involving the Stokes operator. The contents of the paper can be summarized here. In Section 2, a precise statement of the problem along with a full set of assumptions is given and most of the results proved in the paper are stated. Section 3 deals with the continuous dependence of the solution on the data, while Section 4 introduces an approximating problem based on the Moreau–Yosida regularizations of the convex function and on a Faedo–Galerkin scheme, which is sharply discussed about existence of the approximating solution and the proof of a priori estimates. In Section 5 the existence proof is terminated, by taking the limits with respect to the Yosida approximation parameters, and the further regularity results are derived. Section 6 brings the analysis of the long-time behavior and the characterization of the ω -limit as set of stationary solutions to the system (1.1)–(1.2). Finally, Section 7 is completely devoted to the study of the asymptotic behavior of the system (1.1)–(1.2) as the parameter σ tends to 0: the convergence to a phase relaxation problem at the limit is rigorously proved.

2 Statement of the problem and results

In this section, we state precise assumptions and notations and present our results. First of all, the set $\Omega \subset \mathbb{R}^3$ is assumed to be bounded, connected and smooth, and ν and ∂_ν denote the outward unit normal vector field on $\Gamma := \partial\Omega$ and the corresponding normal

derivative, respectively. In order to simplify the notation, we set

$$H := L^2(\Omega) \quad (2.1)$$

and endow H with its standard norm $\|\cdot\|$ and inner product (\cdot, \cdot) . As far as our assumptions are concerned, we first postulate that

$$\begin{aligned} A : D(A) \subset H \rightarrow H \quad \text{and} \quad B : D(B) \subset H \rightarrow H \quad \text{are} \\ \text{unbounded monotone self-adjoint linear operators with compact resolvents} \end{aligned} \quad (2.2)$$

and introduce some function spaces and fractional operators. We avoid the background of interpolation theory and give direct definitions. The above assumption implies that there are sequences $\{\lambda_j\}$ and $\{\mu_j\}$ of eigenvalues and orthonormal sequences $\{e_j\}$ and $\{\eta_j\}$ of corresponding eigenvectors, that is,

$$Ae_j = \lambda_j e_j, \quad B\eta_j = \mu_j \eta_j \quad \text{and} \quad (e_i, e_j) = (\eta_i, \eta_j) = \delta_{ij} \quad \text{for } i, j = 1, 2, \dots \quad (2.3)$$

such that

$$\begin{aligned} 0 \leq \lambda_1 \leq \lambda_2 \leq \dots \quad \text{and} \quad 0 \leq \mu_1 \leq \mu_2 \leq \dots \\ \text{with} \quad \lim_{j \rightarrow \infty} \lambda_j = \lim_{j \rightarrow \infty} \mu_j = +\infty, \end{aligned} \quad (2.4)$$

$$\{e_j\} \text{ and } \{\eta_j\} \text{ are complete systems in } H. \quad (2.5)$$

These assumptions allow us to introduce the Hilbert spaces V_A^r and V_B^σ and the power operators A^r and B^σ (for some arbitrary positive real exponents r and σ) as follows

$$V_A^r := D(A^r) = \left\{ v \in H : \sum_{j=1}^{\infty} |\lambda_j^r(v, e_j)|^2 < +\infty \right\}, \quad (2.6)$$

$$V_B^\sigma := D(B^\sigma) = \left\{ v \in H : \sum_{j=1}^{\infty} |\mu_j^\sigma(v, \eta_j)|^2 < +\infty \right\}, \quad (2.7)$$

$$\begin{aligned} A^r v = \sum_{j=1}^{\infty} \lambda_j^r(v, e_j) e_j \quad \text{and} \quad B^\sigma v = \sum_{j=1}^{\infty} \mu_j^\sigma(v, \eta_j) \eta_j \\ \text{for } v \in V_A^r \text{ and } v \in V_B^\sigma, \text{ respectively.} \end{aligned} \quad (2.8)$$

Note that the series in (2.8) are convergent in the strong topology of H due to the properties of the coefficients. We endow V_A^r and V_B^σ with the graph norms and inner products

$$\|v\|_{A,r}^2 := (v, v)_{A,r} \quad \text{and} \quad (v, w)_{A,r} := (v, w) + (A^r v, A^r w) \quad (2.9)$$

$$\|v\|_{B,\sigma}^2 := (v, v)_{B,\sigma} \quad \text{and} \quad (v, w)_{B,\sigma} := (v, w) + (B^\sigma v, B^\sigma w) \quad (2.10)$$

for $v, w \in V_A^r$ and $v, w \in V_B^\sigma$, respectively. If r_i and σ_i are arbitrary positive exponents, it is clear that

$$(A^{r_1+r_2} v, w) = (A^{r_1} v, A^{r_2} w) \quad \text{for every } v \in V_A^{r_1+r_2} \text{ and } w \in V_A^{r_2}, \quad (2.11)$$

$$(B^{\sigma_1+\sigma_2} v, w) = (B^{\sigma_1} v, B^{\sigma_2} w) \quad \text{for every } v \in V_B^{\sigma_1+\sigma_2} \text{ and } w \in V_B^{\sigma_2}. \quad (2.12)$$

Moreover, since A^r and B^σ are symmetric, for $r, \sigma > 0$ we also have

$$(\partial_t v, A^{2r} v) = \frac{1}{2} \frac{d}{dt} \|A^r v\|^2 \quad \text{and} \quad (\partial_t w, B^{2\sigma} w) = \frac{1}{2} \frac{d}{dt} \|B^\sigma w\|^2 \quad (2.13)$$

for every $v \in H^1(0, T; H) \cap L^2(0, T; V_A^{2r})$ and $w \in H^1(0, T; H) \cap L^2(0, T; V_B^{2\sigma})$, respectively. We also remark that for every $r, \sigma > 0$

$$\text{the embeddings } V_A^r \subset H \text{ and } V_B^\sigma \subset H \text{ are compact,} \quad (2.14)$$

as one immediately sees by using (2.4).

Remark 2.1. Let us mention some simple situation for possible operators A and B . In view of (2.4) and (2.5), a standard elliptic operator with Dirichlet boundary conditions provides an example with a strictly positive first eigenvalue. Another operator that can be considered is the Laplace operator $-\Delta$ with Neumann boundary conditions, which corresponds to the choice $D(-\Delta) = \{v \in H^2(\Omega) : \partial_\nu v = 0\}$; in this case the first eigenvalue is 0 and it is simple, with corresponding eigenfunctions that are the constant functions.

Coming back to our system, we fix r and σ once and for all. Thus, we postulate that

$$r, \sigma \in (0, +\infty). \quad (2.15)$$

For the nonlinearities, we require the properties listed below and notice that they are fulfilled by all of the significant potentials (1.3)–(1.5). We assume that

$$\widehat{\beta} : \mathbb{R} \rightarrow [0, +\infty] \quad \text{is convex, proper and l.s.c. with } \widehat{\beta}(0) = 0, \quad (2.16)$$

$$\widehat{\pi} : \mathbb{R} \rightarrow \mathbb{R} \quad \text{is of class } C^1 \text{ with a Lipschitz continuous first derivative.} \quad (2.17)$$

We set, for convenience,

$$\beta := \partial \widehat{\beta} \quad \text{and} \quad \pi := \widehat{\pi}'. \quad (2.18)$$

Moreover, we term $D(\widehat{\beta})$ and $D(\beta)$ the effective domains of $\widehat{\beta}$ and β , respectively, and, for $r \in D(\beta)$, we use the symbol $\beta^\circ(r)$ for the element of $\beta(r)$ having minimum modulus. We notice that β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. We also remark that (2.17) implies that $\widehat{\pi}$ grows at most quadratically and that π is linearly bounded. Finally, we assume that

$$\ell : \mathbb{R} \rightarrow \mathbb{R} \quad \text{is bounded and Lipschitz continuous.} \quad (2.19)$$

However, in order to keep the operators A^r and B^σ as general as possible, we often assume that ℓ is a constant. Indeed, the more general case (2.19) needs further assumptions.

At this point, we can state the problem we aim to discuss. While the first equation coincides with (1.1), we present (1.2) as a variational inequality written in a weak form on account of (2.12). For the data, we make the following assumptions:

$$f \in L^2(0, T; H) \quad (2.20)$$

$$\vartheta_0 \in V_A^r, \quad \varphi_0 \in V_B^\sigma \quad \text{and} \quad \widehat{\beta}(\varphi_0) \in L^1(\Omega). \quad (2.21)$$

Then, we set

$$Q := \Omega \times (0, T) \quad (2.22)$$

and look for a pair (ϑ, φ) satisfying

$$\vartheta \in H^1(0, T; H) \cap L^\infty(0, T; V_A^r) \cap L^2(0, T; V_A^{2r}), \quad (2.23)$$

$$\varphi \in H^1(0, T; H) \cap L^\infty(0, T; V_B^\sigma), \quad (2.24)$$

$$\widehat{\beta}(\varphi) \in L^1(Q) \quad (2.25)$$

and solving the system

$$\partial_t \vartheta + \ell(\varphi) \partial_t \varphi + A^{2r} \vartheta = f \quad \text{a.e. in } Q, \quad (2.26)$$

$$\begin{aligned} & (\partial_t \varphi(t), \varphi(t) - v) + (B^\sigma \varphi(t), B^\sigma(\varphi(t) - v)) + \int_\Omega \widehat{\beta}(\varphi(t)) + (\pi(\varphi(t)), \varphi(t) - v) \\ & \leq (\vartheta(t) \ell(\varphi(t)), \varphi(t) - v) + \int_\Omega \widehat{\beta}(v) \quad \text{for a.a. } t \in (0, T) \text{ and every } v \in V_B^\sigma, \end{aligned} \quad (2.27)$$

$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \varphi(0) = \varphi_0. \quad (2.28)$$

We notice that equation (2.26) has been written a.e. in Q and in this case the single terms, including $A^{2r} \vartheta$, are interpreted as functions of space and time; another way of reading (2.26) could be a.e. in $(0, T)$ as the equality makes sense for all terms in the space H as well. In our notation, here and in the sequel, we follow the former approach. Concerning (2.27), we warn the reader that

$$\int_\Omega \widehat{\beta}(v) = +\infty \quad \text{whenever} \quad \widehat{\beta}(v) \notin L^1(\Omega).$$

We follow a similar agreement for integrals of the type $\int_Q \widehat{\beta}(v)$ whenever $v \in L^2(Q)$ but $\widehat{\beta}(v) \notin L^1(Q)$. We also remark that (2.27) is equivalent to the following time-integrated version:

$$\begin{aligned} & \int_Q \partial_t \varphi (\varphi - v) + \int_Q B^\sigma \varphi B^\sigma (\varphi - v) + \int_Q \widehat{\beta}(\varphi) + \int_Q \pi(\varphi) (\varphi - v) \\ & \leq \int_Q \vartheta \ell(\varphi) (\varphi - v) + \int_Q \widehat{\beta}(v) \quad \text{for every } v \in L^2(0, T; V_B^\sigma). \end{aligned} \quad (2.29)$$

Remark 2.2. According to the definition of subdifferential (cf., e.g., [10] or [5]), the precise meaning of the inequality (2.27) is that there exists some element $\xi \in L^2(0, T; (V_B^\sigma)^*)$ such that

$$\xi := \vartheta \ell(\varphi) - \partial_t \varphi - B^{2\sigma} \varphi - \pi(\varphi) \in \partial \Phi(y) \quad \text{a.e. in } (0, T),$$

where $\partial \Phi$ is the subdifferential of the convex function $\Phi : V_B^\sigma \rightarrow [0, +\infty]$ defined by

$$\Phi(v) := \int_\Omega \widehat{\beta}(v) \quad \text{if } \widehat{\beta}(v) \in L^1(\Omega), \quad \Phi(v) := +\infty \quad \text{otherwise.}$$

Indeed, we point out that the subdifferential $\partial \Phi$ is a maximal monotone operator from V_B^σ to $(V_B^\sigma)^*$. In this sense, (2.27) turns out to be a slight generalization of (1.2).

The assumptions (2.15)–(2.18) we have made till now on the structure are very general. Nevertheless, they are sufficient to guarantee well-posedness and continuous dependence

at least if ℓ is linear (the nonlinear case is discussed later on). In the result stated below and in the rest of the paper, for $v \in L^1(Q)$, the symbol $1*v$ denotes the function belonging to $L^1(Q)$ that is defined by

$$(1*v)(x, t) := \int_0^t v(x, s) ds \quad \text{for a.a. } (x, t) \in Q. \quad (2.30)$$

Theorem 2.3. *Assume that (2.15)–(2.18) are satisfied and that ℓ is a constant. Moreover, let the assumptions (2.20)–(2.21) on the data be fulfilled. Then there exists a unique pair (ϑ, φ) satisfying (2.23)–(2.25) and solving problem (2.26)–(2.28). Moreover, if $(f_i, \vartheta_{0i}, \varphi_{0i})$, $i = 1, 2$, are two choices of the data and (ϑ_i, φ_i) are the corresponding solutions, then we have*

$$\begin{aligned} & \|\vartheta_1 - \vartheta_2\|_{L^2(0, T; H)} + \|1 * (\vartheta_1 - \vartheta_2)\|_{L^\infty(0, T; V_A^r)} + \|\varphi_1 - \varphi_2\|_{L^\infty(0, T; H) \cap L^2(0, T; V_B^\sigma)} \\ & \leq K (\|1 * (f_1 - f_2)\|_{L^2(0, T; H)} + \|\vartheta_{01} - \vartheta_{02}\| + \|\varphi_{01} - \varphi_{02}\|) \end{aligned} \quad (2.31)$$

with a constant K that depends only on ℓ , some Lipschitz constant for π , and T .

At this point, one can wonder whether both $B^{2\sigma}\varphi$ and $\beta(\varphi)$ make sense in $L^2(Q)$ and (2.27) yields something that is closer to (1.2), like

$$\partial_t \varphi + B^{2\sigma} \varphi + \xi + \pi(\varphi) = \ell \vartheta \quad \text{a.e. in } Q \quad (2.32)$$

for some function ξ on Q satisfying $\xi \in \beta(\varphi)$ a.e. in Q . This depends on the assumption

$$\beta_\varepsilon(v) \in V_B^\sigma \quad \text{and} \quad (B^\sigma \beta_\varepsilon(v), B^\sigma v) \geq 0 \quad \text{for every } v \in V_B^\sigma \text{ and } \varepsilon > 0 \quad (2.33)$$

where β_ε denotes the Yosida regularization of β at the level $\varepsilon > 0$ (see, e.g., [10, p. 28]). We notice that (2.33) does not follow from (2.15)–(2.18) as a consequence and is rather restrictive. Essentially, it is fulfilled whenever $B^{2\sigma}$ is one of the more usual second order linear elliptic operators with boundary conditions of a standard type, indeed. Therefore, in the general case of fractional powers, the more proper formulation of the equation (1.2) for φ is the variational inequality (2.27) (see also Remark 2.2).

Proposition 2.4. *In addition to the assumptions of Theorem 2.3, suppose that (2.33) is fulfilled and let (ϑ, φ) be the solution to (2.26)–(2.28). Then, φ enjoys the regularity property*

$$\varphi \in L^2(0, T; V_B^{2\sigma}) \quad (2.34)$$

and there exists ξ satisfying

$$\xi \in L^2(0, T; H) \quad \text{and} \quad \xi \in \beta(\varphi) \quad \text{a.e. in } Q, \quad (2.35)$$

such that the differential equation (2.32) holds true.

Independently of assumption (2.33) and of the above result, we can show some more regularity for the solution if the datum φ_0 satisfies some proper conditions, as stated below.

Theorem 2.5. *In addition to the assumptions of Theorem 2.3, suppose that*

$$\varphi_0 \in V_B^{2\sigma} \quad \text{and} \quad \beta^\circ(\varphi_0) \in H. \quad (2.36)$$

Then, the solution (ϑ, φ) to problem (2.26)–(2.28) also satisfies

$$\partial_t \varphi \in L^\infty(0, T; H) \cap L^2(0, T; V_B^\sigma). \quad (2.37)$$

Remark 2.6. Of course, to each of our existence and regularity results one can associate a bound for some norm of the solution through a constant that depends only on the assumptions at hand and T . Such bounds are obtained from the construction of the solution, directly. For instance, concerning Theorem 2.3, we have the following estimate

$$\|\vartheta\|_{H^1(0, T; H) \cap L^\infty(0, T; V_A^r) \cap L^2(0, T; V_A^{2r})} + \|\varphi\|_{H^1(0, T; H) \cap L^\infty(0, T; V_B^\sigma)} \leq K_1, \quad (2.38)$$

where the constant K_1 depends only on the structure of the system, the norms of the data corresponding to (2.20)–(2.21), and T .

Concerning the case of a non-constant ℓ that satisfies just (2.19), we can show some results which depend on further assumptions on the operators A^r and B^σ . Namely, we require the following Sobolev-type embeddings:

$$\begin{aligned} &\text{there exist } p, q \in [1, +\infty] \text{ with } \frac{1}{p} + \frac{2}{q} = 1 \text{ such that } V_A^r \subset L^p(\Omega) \text{ and} \\ &V_B^\sigma \subset L^q(\Omega), \text{ the embeddings being continuous and compact, respectively.} \end{aligned} \quad (2.39)$$

We notice that this implies the compactness inequality

$$\|v\|_q \leq \delta \|B^\sigma v\| + C_\delta \|v\| \quad \text{for every } v \in V_B^\sigma \text{ and } \delta > 0, \quad (2.40)$$

with some constant c_δ depending on δ . Of course, $\|\cdot\|_q$ denotes the norm in $L^q(\Omega)$. This notation is also used in the next sections.

Remark 2.7. It is worth noting that assumption (2.39) is satisfied in the case of standard second order operators A and B provided that r and σ are not too small. Assume, for instance, that A and B are the Laplace operators with either Dirichelt or Neumann boundary conditions. Then, for $r, \sigma \in (0, 1)$, the spaces V_A^r and V_B^σ are embedded into the fractional Sobolev spaces $H^{2r}(\Omega)$ and $H^{2\sigma}(\Omega)$, respectively. Therefore, by also assuming $r, \sigma < 3/4$ for simplicity, we see that the three-dimensional embeddings required in (2.39) hold true provided that

$$2r - \frac{3}{2} \geq -\frac{3}{p} \quad \text{and} \quad 2\sigma - \frac{3}{2} > -\frac{3}{q}, \quad \text{or} \quad \frac{1}{p} \geq \frac{1}{2} - \frac{2r}{3} \quad \text{and} \quad \frac{2}{q} > 1 - \frac{4\sigma}{3}.$$

Hence, the existence of p and q as in (2.39) is ensured if in addition it is assumed that

$$\left(\frac{1}{2} - \frac{2r}{3}\right) + \left(1 - \frac{4\sigma}{3}\right) < 1, \quad \text{i.e.,} \quad r + 2\sigma > \frac{3}{4}.$$

Theorem 2.8. *Besides (2.15)–(2.18), assume that (2.19) and (2.39) are fulfilled. Moreover, let the assumptions (2.20)–(2.21) on the data be satisfied. Then, the same conclusions of Theorem 2.3 hold true, but with a constant K also depending on the norms of the data $(f_i, \vartheta_{0i}, \varphi_{0i})$, $i = 1, 2$, in (2.20)–(2.21).*

Proposition 2.9. *In addition to the assumptions of Theorem 2.8, suppose that (2.33) is fulfilled. Then, the same conclusions of Proposition 2.4 hold true.*

Theorem 2.10. *In addition to the assumptions of Theorem 2.8, assume (2.36). Then, the same conclusions of Theorem 2.5 hold true.*

The subsequent aim of this paper is the study of the longtime behavior of the solution. Precisely, under the structural assumptions postulated in one of Theorems 2.3 and 2.8 and the assumptions (2.21) on the initial data, if f is defined on the whole half-line $t \geq 0$ and satisfies (2.20) for every $T > 0$, the existence of a unique global solution defined in $[0, +\infty)$ and satisfying (2.23)–(2.25) for every $T > 0$ is guaranteed. However, in order to treat its longtime behavior, we need further assumptions that do not follow from the other hypotheses and are commented in the next remarks. First of all, we postulate the following coercivity condition

$$\text{there exist some positive constants } \alpha \text{ and } C \text{ such that} \\ \widehat{\beta}_\varepsilon(r) + \widehat{\pi}(r) \geq \alpha r^2 - C \quad \text{for every } r \in \mathbb{R} \text{ and for } \varepsilon > 0 \text{ small enough,} \quad (2.41)$$

where $\widehat{\beta}_\varepsilon$ is the Moreau regularization of $\widehat{\beta}$ at the level ε (see, e.g., [10, Prop. 2.11 p. 39]). If ℓ is a constant, we do not need anything else on the structure of the system. On the contrary, if ℓ just satisfies (2.19), we have to reinforce the condition (2.39) on the operators A^r and B^σ by requiring that

$$\text{there exist } p, q \in [1, +\infty] \text{ with } \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \text{ such that } V_A^r \subset L^p(\Omega) \text{ and} \\ V_B^\sigma \subset L^q(\Omega), \text{ the embeddings being continuous and compact, respectively.} \quad (2.42)$$

Remark 2.11. We notice that (2.41) is satisfied by all of the examples (1.3)–(1.5). More generally, if $\widehat{\pi}(s) = -C_0 s^2$ with $C_0 > 0$ (up to an additive constant) like in the quoted examples, in order that (2.41) holds true it is sufficient to assume that $\widehat{\beta}(s) + \widehat{\pi}(s) \geq 2\alpha s^2 - C$ with the same α and C . Indeed, we have for every $s \in \mathbb{R}$

$$\widehat{\beta}_\varepsilon(s) = \min_{\tau \in \mathbb{R}} \left(\frac{1}{2\varepsilon} (\tau - s)^2 + \widehat{\beta}(\tau) \right) \\ \geq \min_{\tau \in \mathbb{R}} \left(\frac{1}{2\varepsilon} (\tau - s)^2 + (2\alpha + C_0)\tau^2 - C \right) = (2\alpha + C_0 + O(\varepsilon))s^2 - C,$$

whence

$$\widehat{\beta}_\varepsilon(s) + \widehat{\pi}(s) \geq (2\alpha + O(\varepsilon))s^2 - C \geq \alpha s^2 - C$$

if ε is small enough.

We also remark that (2.41) can be weakened by replacing s^2 with $|s|$ on the right-hand side under proper assumptions on the operator B that ensure a Poincaré type inequality (see [17, Prop. 3.1] for a similar situation regarding the operator A). However, we assume (2.41) in order to keep the linear operators as general as possible.

Remark 2.12. In the same framework of Remark 2.7, (2.42) is satisfied if $r + \sigma > 3/4$.

Remark 2.13. We show that (2.42) actually is a reinforcement of (2.39), i.e., that the former implies the latter. Given a choice (p_0, q_0) of (p, q) satisfying (2.42), we construct (p, q) fulfilling (2.39). We take $q = q_0$, so that V_B^σ is compactly embedded in $L^q(\Omega)$; then, we observe that $q_0 \geq 2$ and define $p \in [1, +\infty]$ by means of the equality $1/p = 1 - (2/q_0)$, so that $(1/p) + (2/q) = 1$. Moreover, we have that

$$\frac{1}{p} - \frac{1}{p_0} = \left(1 - \frac{2}{q_0}\right) - \left(\frac{1}{2} - \frac{1}{q_0}\right) = \frac{1}{2} - \frac{1}{q_0} \geq 0$$

whence $p \leq p_0$. Hence the continuous embedding $V_A^r \subset L^{p_0}(\Omega)$ we are assuming implies the continuous embedding $V_A^r \subset L^p(\Omega)$.

Concerning the source term f , we require that it tends to zero in a weak sense as time tends to infinity. Namely, we assume that

$$f \in L^1(0, +\infty; H) \cap L^2(0, +\infty; H). \quad (2.43)$$

Under these assumptions, we study the ω -limit of (ϑ, φ) in the weak topology of $H \times H$. This is defined as follows:

$$\begin{aligned} \omega := \{ & (\vartheta_\omega, \varphi_\omega) \in H \times H : \text{ there exists } \{t_n\} \nearrow +\infty \text{ such that} \\ & (\vartheta(t_n), \varphi(t_n)) \rightarrow (\vartheta_\omega, \varphi_\omega) \text{ weakly in } H \times H \}. \end{aligned} \quad (2.44)$$

We notice that the above definition is meaningful since both ϑ and φ are H -valued continuous functions. However, the ω -limit might be empty. Our results states that this is not the case and that every element of ω is a pair (ϑ_s, φ_s) satisfying

$$\vartheta_s \in V_A^r \quad \text{and} \quad \varphi_s \in V_B^\sigma \quad (2.45)$$

and solving the problem

$$A^r \vartheta_s = 0 \quad \text{a.e. in } \Omega \quad (2.46)$$

$$\begin{aligned} & (B^\sigma \varphi_s, B^\sigma(\varphi_s - v)) + \int_\Omega \widehat{\beta}(\varphi_s) + (\pi(\varphi_s), \varphi_s - v) \\ & \leq (\vartheta_s \ell(\varphi_s), \varphi_s - v) + \int_\Omega \widehat{\beta}(v) \quad \text{for every } v \in V_B^\sigma. \end{aligned} \quad (2.47)$$

We note that this system simply means that (ϑ_s, φ_s) is a stationary solution to problem (2.26)–(2.27), since, given any $r_1 > 0$, in particular $r_1 = 2r$, (2.46) is equivalent to $A^{r_1} \vartheta_s = 0$. Indeed, such equations respectively mean the conditions $\lambda_j^r(\vartheta_s, e_j) = 0$ for every j and $\lambda_j^{r_1}(\vartheta_s, e_j) = 0$ for every j , and the latter conditions are equivalent to each other. However, we keep (2.46) in that form for convenience. The result we state covers both cases regarding ℓ .

Theorem 2.14. *Assume (2.15)–(2.18), (2.41) and either that ℓ is a constant or that (2.42) and (2.19) are fulfilled. Moreover, assume (2.21) on the initial data and (2.43) on the source term, and let (ϑ, φ) satisfy (2.23)–(2.28) for every $T > 0$. Then, the ω -limit (2.44) is nonempty and every element of it is a pair (ϑ_s, φ_s) satisfying (2.45)–(2.47), that is, it is a stationary solution.*

The last set of results is concerned with the asymptotic behavior of our system (2.26)–(2.28) as the coefficient σ of the operator B^σ playing in (2.27) decreases to 0, with the aim of deducing a phase relaxation problem in the limit. These results are obtained in a special situation concerning the data, that is, with ℓ constant and also for a particular choice of the function π (linear case). However, since we recognize that the present section is already rather long and in this setting we need to change a bit the notation for solutions, we prefer to postpone not only the proofs but also the statements for this part of the theory at the last section.

The remainder of the paper is organized as follows. The uniqueness and continuous dependence result is proved in Section 3, while the existence of a solution and its regularity are shown in Section 5 and are prepared by the study of the approximating problem introduced in Section 4. Section 6 is devoted to the longtime behavior of the solution. Finally, Section 7 is concerned with the study of the limiting problem as the exponent σ of the operator B^σ tends to 0.

Throughout the paper, we widely use the notation

$$Q_t := \Omega \times (0, t) \quad \text{for } t \in (0, T], \quad \text{with } Q := Q_T, \quad (2.48)$$

as well as the Hölder inequality and the elementary Young inequality

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \geq 0 \text{ and } \delta > 0 \quad (2.49)$$

and we follow the general rule we explain at once concerning the constants. The small-case italic c without subscripts stands for possibly different constants that may only depend on the operators A^r and B^σ , the shape of the nonlinearities β , π and ℓ , the properties of the data involved in the statements at hand, and the final time T , unless some warning is given in the opposite direction. Thus, the values of such constants do not depend on further parameters (like the regularization parameter ε we introduce in Section 4), and it is clear that they might change from line to line and even in the same formula or chain of inequalities. If δ is any parameter (e.g., ε), the symbol c_δ stands for (possibly different) constants that depend on δ , in addition. In contrast, we use other symbols (e.g., capital letters) for precise values of constants we want to refer to.

3 Uniqueness and continuous dependence

In this section, we prove the uniqueness part of Theorem 2.3 and the continuous dependence estimate. Moreover, we sketch how to modify our argument for the case considered in Theorem 2.8. By noticing that uniqueness follows from (2.31) provided that this is shown for every pair of solutions, we prove just the latter. We fix a pair of data as in the statement and any pair of corresponding solutions and set for convenience

$$\begin{aligned} f &:= f_1 - f_2, & \vartheta_0 &:= \vartheta_{01} - \vartheta_{02}, & \varphi_0 &:= \varphi_{01} - \varphi_{02} \\ \vartheta &:= \vartheta_1 - \vartheta_2 & \text{and} & & \varphi &:= \varphi_1 - \varphi_2. \end{aligned}$$

Assuming that ℓ is a constant, we write (2.26) for both solutions and integrate the difference with respect to time. We obtain

$$\vartheta + \ell\varphi + A^{2r}(1 * \vartheta) = 1 * f + \vartheta_0 + \ell\varphi_0 \quad \text{a.e. in } Q. \quad (3.1)$$

At this point, we multiply the above equality by ϑ and integrate over Q_t , with an arbitrary $t \in (0, T)$. On account of (2.11) and (2.13), we get

$$\begin{aligned} & \int_{Q_t} |\vartheta|^2 + \ell \int_{Q_t} \varphi \vartheta + \frac{1}{2} \|A^r(1 * \vartheta)(t)\|^2 \\ &= \int_{Q_t} (1 * f) \vartheta + \int_{Q_t} (\vartheta_0 + \ell \varphi_0) \vartheta. \end{aligned} \quad (3.2)$$

At the same time, we write (2.27) for both solution and choose $v = \varphi_2(t)$ and $v = \varphi_1(t)$ in the inequalities we obtain, respectively. Then, we sum up and integrate with respect to time. As the contributions involving $\widehat{\beta}$ cancel each other, we obtain

$$\begin{aligned} & \frac{1}{2} \|\varphi(t)\|^2 + \int_0^t \|B^\sigma \varphi(s)\|^2 ds \\ & \leq \frac{1}{2} \|\varphi_0\|^2 - \int_{Q_t} (\pi(\varphi_1) - \pi(\varphi_2)) \varphi + \ell \int_{Q_t} \vartheta \varphi. \end{aligned} \quad (3.3)$$

Now, we add (3.2) to (3.3) and notice that the terms containing ℓ disappear. With the help of the Lipschitz continuity of π (see (2.17)) and of the Young inequality, we deduce that

$$\begin{aligned} & \int_{Q_t} |\vartheta|^2 + \frac{1}{2} \|A^r(1 * \vartheta)(t)\|^2 + \frac{1}{2} \|\varphi(t)\|^2 + \int_0^t \|B^\sigma \varphi(s)\|^2 ds \\ & \leq \frac{1}{2} \int_{Q_t} |\vartheta|^2 + \int_{Q_t} |1 * f|^2 + \int_{Q_t} |\vartheta_0 + \ell \varphi_0|^2 + \frac{1}{2} \|\varphi_0\|^2 + c \int_{Q_t} |\varphi|^2. \end{aligned} \quad (3.4)$$

Thus, (2.31) immediately follows by applying the Gronwall lemma. \square

In the nonlinear case of Theorem 2.8, the equality (3.1) has to be replaced by

$$\begin{aligned} & \vartheta + \widehat{\ell}(\varphi_1) - \widehat{\ell}(\varphi_2) + A^{2r}(1 * \vartheta) = 1 * f + \vartheta_0 + \widehat{\ell}(\varphi_{01}) - \widehat{\ell}(\varphi_{02}) \\ & \text{where } \widehat{\ell}(s) := \int_0^s \ell(\tau) d\tau \text{ for } s \in \mathbb{R}. \end{aligned}$$

Moreover, the last term of (3.3) has to be modified in an obvious way. Hence, the cancellation of the integrals involving ℓ does not occur any longer in summing up, and the main difference with respect to the previous case is the following: as a further contribution to the right-hand side of the final inequality, we have the integral over Q_t of the sum

$$\begin{aligned} & (\widehat{\ell}(\varphi_2) - \widehat{\ell}(\varphi_1)) \vartheta + (\vartheta_1 \ell(\varphi_1) - \vartheta_2 \ell(\varphi_2)) \varphi \\ &= \vartheta_1 \left(\widehat{\ell}(\varphi_2) - \widehat{\ell}(\varphi_1) - \ell(\varphi_1)(\varphi_2 - \varphi_1) \right) + \vartheta_2 \left(\widehat{\ell}(\varphi_1) - \widehat{\ell}(\varphi_2) - \ell(\varphi_2)(\varphi_1 - \varphi_2) \right). \end{aligned}$$

However, this can be treated with the help of our assumptions. We write the Taylor expansion of $\widehat{\ell}$ around any point $s \in \mathbb{R}$ and see that (2.19) implies

$$|\widehat{\ell}(r) - \widehat{\ell}(s) - \ell(s)(r - s)| \leq c |r - s|^2 \text{ for every } r, s \in \mathbb{R}.$$

Hence, we deduce that

$$\int_{Q_t} \left((\widehat{\ell}(\varphi_2) - \widehat{\ell}(\varphi_1)) \vartheta + (\vartheta_1 \ell(\varphi_1) - \vartheta_2 \ell(\varphi_2)) \varphi \right) \leq c \int_{Q_t} (|\vartheta_1| + |\vartheta_2|) |\varphi|^2.$$

At this point, we invoke (2.39) and apply (2.40). Thus, we can estimate the right-hand side of the above inequality and obtain

$$\begin{aligned}
& \int_{Q_t} \left((\widehat{\ell}(\varphi_2) - \widehat{\ell}(\varphi_1)) \vartheta + (\vartheta_1 \ell(\varphi_1) - \vartheta_2 \ell(\varphi_2)) \varphi \right) \\
& \leq c \int_0^t \left(\|\vartheta_1(s)\| + \|\vartheta_2(s)\| \right) \|\varphi(s)\|_q^2 ds \\
& \leq c \left(\|\vartheta_1\|_{L^\infty(0,T;V_A^r)} + \|\vartheta_2\|_{L^\infty(0,T;V_A^r)} \right) \int_0^t \|\varphi(s)\|_q^2 ds \\
& \leq \frac{1}{2} \int_0^t \|B^\sigma \varphi(s)\|^2 ds + c \int_0^t \|\varphi(s)\|^2 ds
\end{aligned}$$

where the last value of c also depends on the norms of ϑ_1 and ϑ_2 just written. Therefore, we can come back to the modified (3.4) and conclude as in the previous proof by applying the Gronwall lemma.

4 Approximation

In this section, we prepare some auxiliary material that will be used to perform the proofs of the existence parts of Theorems 2.3 and 2.8 of the next section. We introduce an approximating problem by fixing $\varepsilon > 0$ and replacing the function $\widehat{\beta}$ and its subdifferential β by their Moreau-Yosida regularizations $\widehat{\beta}_\varepsilon$ and β_ε at the level ε (see, e.g., [10, p. 28 and Prop. 2.11 p. 39]). Thus, β_ε is monotone and Lipschitz continuous and coincides with the derivative of $\widehat{\beta}_\varepsilon$. Moreover, by also accounting for (2.16), it holds that

$$0 \leq \widehat{\beta}_\varepsilon(r) \leq \widehat{\beta}(r), \quad \widehat{\beta}_\varepsilon(r) \leq \widehat{\beta}_{\varepsilon'}(r) \quad \text{if } \varepsilon' < \varepsilon, \quad \text{and} \quad \lim_{\varepsilon \searrow 0} \widehat{\beta}_\varepsilon(r) = \widehat{\beta}(r) \quad (4.1)$$

for every $r \in \mathbb{R}$. We set for convenience

$$F_\varepsilon := \widehat{\beta}_\varepsilon + \widehat{\pi}, \quad \text{whence} \quad F'_\varepsilon = \beta_\varepsilon + \pi. \quad (4.2)$$

Hence, the approximating problem consists in finding a pair $(\vartheta_\varepsilon, \varphi_\varepsilon)$ satisfying

$$\vartheta_\varepsilon \in H^1(0, T; H) \cap L^\infty(0, T; V_A^r) \cap L^2(0, T; V_A^{2r}) \quad (4.3)$$

$$\varphi_\varepsilon \in H^1(0, T; H) \cap L^\infty(0, T; V_B^\sigma) \cap L^2(0, T; V_B^{2\sigma}) \quad (4.4)$$

and solving the system

$$\partial_t \vartheta_\varepsilon + \ell(\varphi_\varepsilon) \partial_t \varphi_\varepsilon + A^{2r} \vartheta_\varepsilon = f \quad \text{a.e. in } Q \quad (4.5)$$

$$\partial_t \varphi_\varepsilon + B^{2\sigma} \varphi_\varepsilon + F'_\varepsilon(\varphi_\varepsilon) = \vartheta_\varepsilon \ell(\varphi_\varepsilon) \quad \text{a.e. in } Q \quad (4.6)$$

$$\vartheta_\varepsilon(0) = \vartheta_0 \quad \text{and} \quad \varphi_\varepsilon(0) = \varphi_0. \quad (4.7)$$

Notice that we have approximated the strong form (1.2) rather than (2.27). The aim of this section is to prove that the above problem is well-posed. We first treat the case that ℓ is a constant. The more general situation is considered later on in the section.

Theorem 4.1. *Under the assumptions of Theorem 2.3, the approximating problem (4.5)–(4.7) has a unique solution $(\vartheta_\varepsilon, \varphi_\varepsilon)$ satisfying (4.3)–(4.4).*

As far as uniqueness is concerned, it suffices to observe that (4.6) implies the analogue of (2.27) obtained by replacing $\widehat{\beta}$ by $\widehat{\beta}_\varepsilon$ and that $\widehat{\beta}_\varepsilon$ satisfies (2.16). Thus, we can apply what we have just established in Section 3. In order to prove the existence part, we use a Faedo-Galerkin scheme depending on the parameter $n \in \mathbb{N}$ and then we let n tend to infinity. By recalling (2.3)–(2.5), we introduce the subspaces

$$V_{A,n} := \text{span}\{e_1, \dots, e_n\} \quad \text{and} \quad V_{B,n} := \text{span}\{\eta_1, \dots, \eta_n\} \quad (4.8)$$

and look for a pair $(\vartheta^n, \varphi^n) \in H^1(0, T; V_{A,n} \times V_{B,n})$ satisfying

$$\begin{aligned} (\partial_t \vartheta^n + \ell \partial_t \varphi^n + A^{2r} \vartheta^n, v) &= (f, v) \\ \text{a.e. in } (0, T) \text{ for every } v \in V_{A,n} \end{aligned} \quad (4.9)$$

$$\begin{aligned} (\partial_t \varphi^n + B^{2\sigma} \varphi^n + F'_\varepsilon(\varphi^n), v) &= \ell(\vartheta^n, v) \\ \text{a.e. in } (0, T) \text{ for every } v \in V_{B,n} \end{aligned} \quad (4.10)$$

$$\begin{aligned} (\vartheta^n(0), v) &= (\vartheta_0, v) \quad \text{and} \quad (\varphi^n(0), v) = (\varphi_0, v) \\ \text{for every } v \in V_{A,n} \text{ and } v \in V_{B,n}, \text{ respectively.} \end{aligned} \quad (4.11)$$

Since ε is fixed at the moment, we did not stress the dependence of (ϑ^n, φ^n) on ε in the notation. First of all, we establish the existence of a global solution to the above problem. We represent (ϑ^n, φ^n) in terms of the bases of $V_{A,n}$ and $V_{B,n}$ as follows:

$$\vartheta^n(t) = \sum_{j=1}^n \vartheta_j^n(t) e_j \quad \text{and} \quad \varphi^n(t) = \sum_{j=1}^n \varphi_j^n(t) \eta_j$$

where the functions ϑ_j^n and φ_j^n are looked for in $H^1(0, T; \mathbb{R})$. If we equivalently let $v = e_i$ in (4.9) and $v = \eta_i$ in (4.10) with $i = 1, \dots, n$, we see that the system (4.9)–(4.10) becomes

$$\begin{aligned} \left(\sum_{j=1}^n \partial_t \vartheta_j^n e_j + \ell \sum_{j=1}^n \partial_t \varphi_j^n \eta_j + \sum_{j=1}^n \lambda_j^{2r} \vartheta_j^n e_j, e_i \right) &= (f, e_i) \\ \text{a.e. in } (0, T) \text{ for } i = 1, \dots, n \\ \left(\sum_{j=1}^n \partial_t \varphi_j^n \eta_j + \sum_{j=1}^n \mu_j^{2\sigma} \varphi_j^n \eta_j + F'_\varepsilon(\sum_{j=1}^n \varphi_j^n \eta_j), \eta_i \right) &= \ell \left(\sum_{j=1}^n \vartheta_j^n e_j, \eta_i \right) \\ \text{a.e. in } (0, T) \text{ for } i = 1, \dots, n. \end{aligned}$$

So, by introducing the n -column vectors $\Theta := {}^t[\vartheta_1^n, \dots, \vartheta_n^n]$ and $\Phi := {}^t[\varphi_1^n, \dots, \varphi_n^n]$, we obtain the following compact form of the system

$$\begin{aligned} \Theta' + \mathcal{E}\Phi' + \Lambda\Theta &= g \quad \text{and} \quad \Phi' + M\Phi + \mathcal{F}(\Theta, \Phi) = 0 \quad \text{or} \\ \Theta' - \mathcal{E}(M\Phi + \mathcal{F}(\Theta, \Phi)) + \Lambda\Theta &= g \quad \text{and} \quad \Phi' + M\Phi + \mathcal{F}(\Theta, \Phi) = 0 \end{aligned} \quad (4.12)$$

where the matrices \mathcal{E} , Λ and M and the functions $g \in L^2(0, T; \mathbb{R}^n)$ and $\mathcal{F} : (\mathbb{R}^n)^2 \rightarrow \mathbb{R}^n$

are defined by

$$\begin{aligned} \mathcal{E} &:= \ell [(\eta_j, e_i)]_{i,j=1,\dots,n}, \quad \Lambda := \text{diag}(\lambda_1^{2r}, \dots, \lambda_n^{2r}), \quad M := \text{diag}(\mu_1^{2\sigma}, \dots, \mu_n^{2\sigma}) \\ g(t) &:= [(f(t), e_i)]_{i=1,\dots,n} \quad \text{for a.a. } t \in (0, T) \\ \mathcal{F}(r, s) &:= \left[\left(F'_\varepsilon \left(\sum_{k=1}^n s_k \eta_k \right) - \ell \sum_{j=1}^n r_j e_j, \eta_i \right) \right]_{i=1,\dots,n} \\ &\text{for } r = (r_1, \dots, r_n) \in \mathbb{R}^n \text{ and } s = (s_1, \dots, s_n) \in \mathbb{R}^n. \end{aligned}$$

Since the Lipschitz continuity of F'_ε (see (4.2)) implies the same property for \mathcal{F} and the function g belongs to $L^2(0, T; \mathbb{R}^n)$, every Cauchy problem for (4.12) has a unique global solution $(\Theta, \Phi) \in H^1(0, T; \mathbb{R}^n \times \mathbb{R}^n)$. On the other hand, (4.11) yields an initial condition for (Θ, Φ) . Therefore, coming back to problem (4.9)–(4.11), we conclude that it has a unique solution $(\vartheta^n, \varphi^n) \in H^1(0, T; V_{A,n} \times V_{B,n})$.

At this point, we are ready to prove the existence part of Theorem 4.1. This will be done by performing a number of a priori estimates and passing to the limit by compactness arguments.

First a priori estimate. We test (4.9) written at the time s by $v = \vartheta^n(s)$ and integrate with respect to s over $(0, t)$, with an arbitrary $t \in (0, T)$. In the same way, we test (4.10) by $\partial_t \varphi^n$, integrate with respect to time and add the same quantity $\int_{Q_t} \varphi^n \partial_t \varphi^n$ to both sides. Then, we sum up and observe that the terms involving ℓ cancel each other. Hence, on account of (2.11)–(2.13), we obtain

$$\begin{aligned} &\frac{1}{2} \|\vartheta^n(t)\|^2 + \int_0^t \|A^r \vartheta^n(s)\|^2 ds + \int_0^t \|\partial_t \varphi^n(s)\|^2 ds + \frac{1}{2} \|\varphi^n(t)\|_{B,\sigma}^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi^n(t)) \\ &= \frac{1}{2} \|\vartheta^n(0)\|^2 + \frac{1}{2} \|B^\sigma \varphi^n(0)\|^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi^n(0)) \\ &+ \int_0^t (f(s), \vartheta^n(s)) ds + \int_0^t (\varphi^n(s) - \pi(\varphi^n(s)), \partial_t \varphi^n(s)) ds. \end{aligned} \quad (4.13)$$

By also recalling (4.1), we see that all the terms on the left-hand side of (4.13) are nonnegative. The sum of the last two integrals on the right-hand side is estimated, owing to assumption (2.17) and the Young inequality, as follows:

$$\begin{aligned} &\int_0^t (f(s), \vartheta^n(s)) ds + \int_0^t (\varphi^n(s) - \pi(\varphi^n(s)), \partial_t \varphi^n(s)) ds \\ &\leq \|f\|_{L^2(0,T;H)}^2 + \int_0^t \|\vartheta^n(s)\|^2 ds + \frac{1}{2} \int_0^t \|\partial_t \varphi^n(s)\|^2 ds + c \int_0^t \|\varphi^n(s)\|^2 ds + c. \end{aligned}$$

Concerning the other terms, we observe that $\vartheta^n(0)$ and $\varphi^n(0)$ are the H -projections of ϑ_0 and φ_0 on $V_{A,n}$ and $V_{B,n}$, respectively, due to (4.11). By also accounting for the Lipschitz continuity of β_ε , we obtain for two of them

$$\|\vartheta^n(0)\|^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi^n(0)) \leq \|\vartheta_0\|^2 + c_\varepsilon (\|\varphi^n(0)\|^2 + 1) \leq \|\vartheta_0\|^2 + c_\varepsilon (\|\varphi_0\|^2 + 1) \leq c_\varepsilon.$$

Finally, we have that

$$B^\sigma \varphi^n(0) = B^\sigma \sum_{j=1}^n (\varphi_0, \eta_j) \eta_j = \sum_{j=1}^n \mu_j^\sigma (\varphi_0, \eta_j) \eta_j$$

whence also

$$\|B^\sigma \varphi^n(0)\|^2 = \sum_{j=1}^n |\mu_j^\sigma(\varphi_0, \eta_j)|^2 \leq \sum_{j=1}^{\infty} |\mu_j^\sigma(\varphi_0, \eta_j)|^2 = \|B^\sigma \varphi_0\|^2.$$

Thus, coming back to (4.13) and applying the Gronwall lemma, we conclude that

$$\|\vartheta^n\|_{L^\infty(0,T;H) \cap L^2(0,T;V_A^r)} + \|\varphi^n\|_{H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma)} \leq c_\varepsilon. \quad (4.14)$$

Second a priori estimate. We test (4.9) by $\partial_t \vartheta^n$ and integrate with respect to time as before. Thanks to (2.11)–(2.13) once more, we obtain

$$\int_0^t \|\partial_t \vartheta^n(s)\|^2 ds + \frac{1}{2} \|A^r \vartheta^n(t)\|^2 = \frac{1}{2} \|A^r \vartheta^n(0)\|^2 + \int_0^t (f(s) - \ell \partial_t \varphi^n(s), \partial_t \vartheta^n(s)) ds.$$

By arguing as before in order to estimate the first term on the right-hand side and using the Young inequality and (4.14) for the second one, we immediately conclude that

$$\|\vartheta^n\|_{H^1(0,T;H) \cap L^\infty(0,T;V_A^r)} \leq c_\varepsilon. \quad (4.15)$$

Limit. By (4.14)–(4.15) and standard weak star compactness results, we have for a (not relabeled) subsequence

$$\vartheta^n \rightharpoonup \vartheta_\varepsilon \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V_A^r), \quad (4.16)$$

$$\varphi^n \rightharpoonup \varphi_\varepsilon \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma). \quad (4.17)$$

In view of the compact embeddings (2.14) and applying a proper strong compactness result (see, e.g., [35, Sect. 8, Cor. 4]), we deduce that

$$\vartheta^n \rightarrow \vartheta_\varepsilon \quad \text{and} \quad \varphi^n \rightarrow \varphi_\varepsilon \quad \text{strongly in } C^0([0,T];H). \quad (4.18)$$

This implies, in particular, that $F'_\varepsilon(\varphi^n)$ converges to $F'_\varepsilon(\varphi_\varepsilon)$ in the same topology, just by Lipschitz continuity. We want to deduce that the following integrated version of the approximating problem

$$\int_0^T (\partial_t(\vartheta_\varepsilon + \ell \varphi_\varepsilon)(s) - f(s), v(s)) ds + \int_0^T (A^r \vartheta_\varepsilon(s), A^r v(s)) ds = 0, \quad (4.19)$$

$$\int_0^T (\partial_t \varphi_\varepsilon(s) + F'_\varepsilon(\varphi_\varepsilon(s)) - \ell \vartheta_\varepsilon(s), v(s)) ds + \int_0^T (B^\sigma \vartheta_\varepsilon(s), B^\sigma v(s)) ds = 0 \quad (4.20)$$

is fulfilled for every $v \in L^2(0,T;V_A^r)$ and every $v \in L^2(0,T;V_B^\sigma)$, respectively. We start from the following integrated versions of the equations (4.9) and (4.10):

$$\int_0^T (\partial_t(\vartheta^n + \ell \varphi^n)(s) - f(s), v(s)) ds + \int_0^T (A^r \vartheta^n(s), A^r v(s)) ds = 0, \quad (4.21)$$

$$\int_0^T (\partial_t \varphi^n(s) + F'_\varepsilon(\varphi^n(s)) - \ell \vartheta^n(s), v(s)) ds + \int_0^T (B^\sigma \vartheta^n(s), B^\sigma v(s)) ds = 0, \quad (4.22)$$

which obviously hold for every $v \in L^2(0, T; V_{A,n})$ and every $v \in L^2(0, T; V_{B,n})$, respectively, due to (2.11)–(2.12). We fix $m \in \mathbb{N}$ and take any $v \in L^2(0, T; V_{A,m})$. For every $n \geq m$, we have that $V_{A,m} \subset V_{A,n}$, whence (4.21) holds for v . By arguing similarly for (4.22) and then letting n tend to infinity, we deduce that (4.19)–(4.20) are satisfied for every $v \in L^2(0, T; V_{A,m})$ and $v \in L^2(0, T; V_{B,m})$, respectively. By a simple density argument, we conclude that the same equations hold true for every $v \in L^2(0, T; V_A^r)$ and every $v \in L^2(0, T; V_B^\sigma)$, respectively. We deduce that $(\vartheta_\varepsilon, \varphi_\varepsilon)$ solves the equivalent problem

$$(\partial_t(\vartheta_\varepsilon + \ell\varphi_\varepsilon) - f, v) + (A^r\vartheta_\varepsilon, A^r v) = 0 \quad \text{a.e. in } (0, T), \text{ for every } v \in V_A^r, \quad (4.23)$$

$$(\partial_t\varphi_\varepsilon + F'_\varepsilon(\varphi_\varepsilon) - \ell\vartheta_\varepsilon, v) + (B^\sigma\vartheta_\varepsilon, B^\sigma v) = 0 \quad \text{a.e. in } (0, T), \text{ for every } v \in V_B^\sigma, \quad (4.24)$$

so that (4.5)–(4.6) follow from the lemma given below. Finally, as (4.18) implies that $\vartheta^n(0)$ and $\varphi^n(0)$ converge to $\vartheta_\varepsilon(0)$ and $\varphi_\varepsilon(0)$ strongly in H , we see that the initial conditions (4.7) follow from the theory of orthogonal projections, and the proof of Theorem 4.1 is complete. \square

Lemma 4.2. *Assume that $u \in V_A^r$ and $\psi \in H$ satisfy*

$$(A^r u, A^r v) = (\psi, v) \quad \text{for every } v \in V_A^r. \quad (4.25)$$

Then, we have that

$$u \in V_A^{2r} \quad \text{and} \quad A^{2r} u = \psi. \quad (4.26)$$

Moreover, the same result hold if A and r are replaced by B and σ , respectively.

Proof. We have that

$$u = \sum_{j=1}^{\infty} (u, e_j) e_j, \quad \psi = \sum_{j=1}^{\infty} (\psi, e_j) e_j \quad \text{and} \quad \sum_{j=1}^{\infty} |\lambda_j^r(u, e_j)|^2 + \sum_{j=1}^{\infty} |(\psi, e_j)|^2 < +\infty.$$

Moreover, (4.25) written with $v = e_i$ implies that $\lambda_i^{2r}(u, e_i) = (\psi, e_i)$ for every i , whence also $\sum_{j=1}^{\infty} |\lambda_j^{2r}(u, e_j)|^2 < +\infty$. Thus, both conditions (4.26) immediately follow. \square

Now, we consider the case of a nonlinear function ℓ . To this concern, we have the analogue of Theorem 4.1, namely

Theorem 4.3. *Under the assumptions of Theorem 2.8, the approximating problem (4.5)–(4.7) has a unique solution $(\vartheta_\varepsilon, \varphi_\varepsilon)$ satisfying (4.3)–(4.4).*

Proof. The argument follows the same line of the previous proof and we just stress the modifications that are needed. Concerning the choice of the Faedo-Galerkin scheme, it suffices to replace ℓ by $\ell(\varphi^n)$ in equations (4.9) and (4.10). However, the existence of a global discrete solution is no longer clear. Indeed, the corresponding system of ordinary differential equations still takes the form (4.12), but \mathcal{E} has to be replaced by $\mathcal{E}(\Phi)$, i.e., by a function depending on Φ , and the definition of \mathcal{F} has to be modified. Precisely, we

have to set

$$\begin{aligned} \mathcal{E}(s) &:= \left[\left(\ell \left(\sum_{j=1}^n s_k \eta_k \right) \eta_j, e_i \right) \right]_{i,j=1,\dots,n} \\ \mathcal{F}(r, s) &:= \left[\left(F'_\varepsilon \left(\sum_{k=1}^n s_k \eta_k \right) - \ell \left(\sum_{k=1}^n s_k \eta_k \right) \sum_{j=1}^n r_j e_j, \eta_i \right) \right]_{i=1,\dots,n} \\ &\text{for } r = (r_1, \dots, r_n) \in \mathbb{R}^n \text{ and } s = (s_1, \dots, s_n) \in \mathbb{R}^n. \end{aligned}$$

Thus, if ℓ' does not vanish identically, the derivatives of \mathcal{F} with respect to s_j have a linear growth with respect to r and the Lipschitz condition that is needed to ensure the existence of a global solution fails. For this reason, we can only conclude that the system has a unique maximal solution (Θ, Φ) defined in some interval $[0, T_n) \subseteq [0, T]$. Thus, in principle, the maximal solution (ϑ^n, φ^n) to the discrete problem (4.9)–(4.11) exists and is defined in the same interval. However, the estimates we can perform show that, for every n , (ϑ^n, φ^n) is bounded in $L^\infty(0, T; V_{A,n} \times V_{B,n})$ by a constant that depends on T but not on T_n . Thus, the same happens for $(\Theta, \Phi) \in L^\infty(0, T; \mathbb{R}^n \times \mathbb{R}^n)$, so that the general theory of ordinary differential equations ensures the existence of a solution in the whole interval $[0, T]$.

Coming to the estimates, the first one goes exactly as we did to obtain (4.14). Indeed, by testing the equations in the same way, the terms involving ℓ cancel each other also in the nonlinear case. Concerning the second estimate, we only have to recall that ℓ is bounded, so that $\ell(\varphi^n) \partial_t \varphi^n$ is bounded in $L^2(0, T; H)$ exactly as $\ell \partial_t \varphi^n$ was before. So, the analogue of (4.15) follows. Furthermore, in taking the limit as n tends to infinity, it suffices to add to the above argument the following convergence property: $\ell(\varphi^n)$ converges to $\ell(\varphi_\varepsilon)$ strongly in $C^0([0, T]; H)$, as a consequence of (4.18) and the Lipschitz continuity of ℓ . Recalling (4.17) as well, it is easy to see that $\ell(\varphi^n) \partial_t \varphi^n$ and $\ell(\varphi^n) \vartheta^n$ are bounded in $L^2(0, T; H)$ and converge weakly in $(L^2(0, T; L^1(\Omega)))$, thus in $L^2(0, T; H)$ to $\ell(\varphi_\varepsilon) \partial_t \varphi_\varepsilon$ and $\ell(\varphi_\varepsilon) \vartheta_\varepsilon$, respectively, so that we can take the limit of the corresponding terms in the analogues of (4.21) and (4.22). Finally, Lemma 4.2 applies also in the present case. This completes the proof. \square

5 Existence and regularity

In this section, we conclude the proofs of the existence and regularity results stated in Section 2. In order to establish the existence parts of both Theorems 2.3 and 2.8, we assume the general hypothesis (2.19) and avoid using (2.39). We start from the approximating problem (4.5)–(4.7) (whose unique solution exists thanks to Theorems 4.1 and 4.3) and perform some a priori estimates.

Uniform estimates. We multiply (4.5) and (4.6) by ϑ_ε and $\partial_t \varphi_\varepsilon$, respectively, sum up and integrate over Q_t , with an arbitrary $t \in (0, T)$. We notice that the terms involving ℓ

cancel each other and add the same term $\int_{Q_t} \varphi_\varepsilon \partial_t \varphi_\varepsilon$ to both sides. We obtain

$$\begin{aligned} & \frac{1}{2} \|\vartheta_\varepsilon(t)\|^2 + \int_0^t \|A^r \vartheta_\varepsilon(s)\|^2 ds + \int_{Q_t} |\partial_t \varphi_\varepsilon|^2 + \frac{1}{2} \|\varphi_\varepsilon(t)\|_{B,\sigma}^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_\varepsilon(t)) \\ &= \frac{1}{2} \|\vartheta_0\|^2 + \frac{1}{2} \|\varphi_0\|_{B,\sigma}^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_0) + \int_{Q_t} (f + \varphi_\varepsilon - \pi(\varphi_\varepsilon)) \partial_t \varphi_\varepsilon. \end{aligned} \quad (5.1)$$

We treat the right-hand side by first using the Young inequality and (4.1), then invoking the assumptions (2.20)–(2.21) and the linear growth of π . Hence, by applying the Gronwall lemma, we infer that

$$\|\vartheta_\varepsilon\|_{L^\infty(0,T;H) \cap L^2(0,T;V_A^r)} + \|\varphi_\varepsilon\|_{H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma)} + \|\widehat{\beta}_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \leq c. \quad (5.2)$$

Next, we multiply (4.5) by $\partial_t \vartheta_\varepsilon$ and integrate over Q_t as before. We obtain

$$\int_{Q_t} |\partial_t \vartheta_\varepsilon|^2 + \frac{1}{2} \|A^r \vartheta_\varepsilon(t)\|^2 = \frac{1}{2} \|A^r \vartheta_0\|^2 + \int_{Q_t} (f - \ell(\varphi_\varepsilon) \partial_t \varphi_\varepsilon) \partial_t \vartheta_\varepsilon.$$

With the help of the boundedness of ℓ , the Young inequality and (5.2), we conclude that

$$\|\vartheta_\varepsilon\|_{H^1(0,T;H) \cap L^\infty(0,T;V_A^r)} \leq c. \quad (5.3)$$

Limit. As in the previous section, we owe to weak star and strong compactness results and deduce from (5.2)–(5.3) that (at least for a subsequence)

$$\begin{aligned} \vartheta_\varepsilon &\rightarrow \vartheta && \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V_A^r) \\ &&& \text{and strongly in } C^0([0,T];H), \end{aligned} \quad (5.4)$$

$$\begin{aligned} \varphi_\varepsilon &\rightarrow \varphi && \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V_B^\sigma) \\ &&& \text{and strongly in } C^0([0,T];H). \end{aligned} \quad (5.5)$$

We deduce that the initial conditions (2.28) hold true and that $\pi(\varphi_\varepsilon) \rightarrow \pi(\varphi)$ and $\ell(\varphi_\varepsilon) \rightarrow \ell(\varphi)$ strongly in $C^0([0,T];H)$ by Lipschitz continuity. It follows that $\ell(\varphi_\varepsilon) \partial_t \varphi_\varepsilon \rightarrow \ell(\varphi) \partial_t \varphi$ and $\ell(\varphi_\varepsilon) \vartheta_\varepsilon \rightarrow \ell(\varphi) \vartheta$ weakly in $L^2(0,T;H)$ (by boundedness in $L^2(0,T;H)$ and weak convergence in $L^2(0,T;L^1(\Omega))$). At this point, it is straightforward to show that (ϑ, φ) satisfies the integrated version of (2.26) similar to (4.19). Then, it also satisfies the analog of (4.23), so that both the full regularity (2.23) and the equation (2.26) follow from Lemma 4.2. So, it remains to prove that φ also satisfies (2.25) and that (ϑ, φ) solves the variational inequality (2.27). We recall that all the above convergence properties hold for some subsequence $\varepsilon_n \searrow 0$, which we can assume to be strictly decreasing without loss of generality. Moreover, we can assume that φ_{ε_n} converges to φ a.e. in Q . We first prove that

$$\int_Q \widehat{\beta}(\varphi) \leq \liminf_{n \rightarrow \infty} \int_Q \widehat{\beta}_{\varepsilon_n}(\varphi_{\varepsilon_n}). \quad (5.6)$$

We take arbitrary indices n and m with $n > m$. Then, $\varepsilon_n < \varepsilon_m$ and we can apply (4.1). We deduce that

$$\widehat{\beta}_{\varepsilon_m}(\varphi_{\varepsilon_n}) \leq \widehat{\beta}_{\varepsilon_n}(\varphi_{\varepsilon_n}) \quad \text{a.e. in } Q, \text{ for every } n > m.$$

Since $\widehat{\beta}_{\varepsilon_m}$ is (Lipschitz) continuous, we thus have that

$$\widehat{\beta}_{\varepsilon_m}(\varphi) = \lim_{n \rightarrow \infty} \widehat{\beta}_{\varepsilon_m}(\varphi_{\varepsilon_n}) = \liminf_{n \rightarrow \infty} \widehat{\beta}_{\varepsilon_m}(\varphi_{\varepsilon_n}) \leq \liminf_{n \rightarrow \infty} \widehat{\beta}_{\varepsilon_n}(\varphi_{\varepsilon_n}) \quad \text{a.e. in } Q.$$

On the other hand, the last condition in (4.1) implies that

$$\widehat{\beta}(\varphi) = \lim_{m \rightarrow \infty} \widehat{\beta}_{\varepsilon_m}(\varphi) \quad \text{a.e. in } Q.$$

Therefore, (5.6) follows from the Fatou lemma. At this point, we can easily conclude. From one side, (5.6) implies (2.25) due to (5.2). On the other hand, it is easily seen that (4.6) implies the analogue of (2.27), i.e., the variational inequality obtained by replacing $\widehat{\beta}$, ϑ and φ in (2.27) by $\widehat{\beta}_\varepsilon$, ϑ_ε and φ_ε , respectively. Therefore, by also combining the strong and weak convergence properties (5.4)–(5.5) and writing ε instead of ε_n for simplicity, we have for every $v \in L^2(0, T; V_B^\sigma)$

$$\begin{aligned} & \int_Q \widehat{\beta}(\varphi) + \int_0^T (B^\sigma \varphi(t), B^\sigma(\varphi(t) - v(t))) dt \\ & \leq \liminf_{\varepsilon \searrow 0} \int_Q \widehat{\beta}_\varepsilon(\varphi_\varepsilon) + \liminf_{\varepsilon \searrow 0} \int_0^T (B^\sigma \varphi_\varepsilon(t), B^\sigma(\varphi_\varepsilon(t) - v(t))) dt \\ & \leq \liminf_{\varepsilon \searrow 0} \left(\int_Q \widehat{\beta}_\varepsilon(\varphi_\varepsilon) + \int_0^T (B^\sigma \varphi_\varepsilon(t), B^\sigma(\varphi_\varepsilon(t) - v(t))) dt \right) \\ & \leq \liminf_{\varepsilon \searrow 0} \left(\int_Q (\ell(\varphi_\varepsilon) \vartheta_\varepsilon - \partial_t \varphi_\varepsilon - \pi(\varphi_\varepsilon))(\varphi_\varepsilon - v) + \int_Q \widehat{\beta}_\varepsilon(v) \right) \\ & = \int_Q (\ell(\varphi) \vartheta - \partial_t \varphi - \pi(\varphi))(\varphi - v) + \int_Q \widehat{\beta}(v) \end{aligned}$$

In the last equality we have used the last (4.1) as well. Hence, (2.27) is proved and the proof is complete. \square

Proofs of Propositions 2.4 and 2.9. Since we use just the boundedness of ℓ , the same proof holds for both propositions. We start from the approximating problem once more and, for a.a. $t \in (0, T)$, we write (4.6) at the time t in the form

$$(B^\sigma \varphi_\varepsilon(t), B^\sigma v) + \int_\Omega \beta_\varepsilon(\varphi_\varepsilon(t)) v = (\vartheta_\varepsilon(t) \ell(\varphi_\varepsilon(t)) - \partial_t \varphi_\varepsilon(t) - \pi(\varphi_\varepsilon(t)), v) \quad \text{for every } v \in V_B^\sigma.$$

Thanks to (2.33), we can choose $v = \beta_\varepsilon(\varphi_\varepsilon(t))$ and obtain

$$\int_\Omega |\beta_\varepsilon(\varphi_\varepsilon(t))|^2 \leq (\vartheta_\varepsilon(t) \ell(\varphi_\varepsilon(t)) - \partial_t \varphi_\varepsilon(t) - \pi(\varphi_\varepsilon(t)), \beta_\varepsilon(\varphi_\varepsilon(t))).$$

Then, the Young inequality, the boundedness of ℓ and (5.2) immediately yield

$$\|\beta_\varepsilon(\varphi_\varepsilon)\|_{L^2(0, T; H)} \leq c.$$

As a consequence, we also have from (4.6) that

$$\|B^{2\sigma} \varphi_\varepsilon\|_{L^2(0, T; H)} \leq c.$$

Therefore, coming back to the proof of Theorems 2.3 and 2.8 just performed, we see that we can add to (5.4)–(5.5) the further convergence properties

$$\beta_\varepsilon(\varphi_\varepsilon) \rightarrow \xi \quad \text{and} \quad B^{2\sigma}\varphi_\varepsilon \rightarrow B^{2\sigma}\varphi \quad \text{weakly in } L^2(0, T; H). \quad (5.7)$$

At this point, it is clear that $\varphi \in L^2(0, T; V_B^{2\sigma})$ and that (2.32) holds true. In order to check the inclusion property in (2.35), it suffices to recall the strong convergence (5.5) of φ_ε and use the maximal monotonicity of β by applying, e.g., [5, Lemma 2.3, p. 38]. \square

Finally, we prove Theorems 2.5 and 2.10. We start from the first of them, i.e., we assume that ℓ is a constant. Clearly, it suffices to establish the estimates corresponding to (2.37) on the solution $(\vartheta_\varepsilon, \varphi_\varepsilon)$ to the approximating problem (4.5)–(4.7).

Regularity estimate. We proceed formally, for brevity. We differentiate (4.6) with respect to time and test the resulting equality by $\partial_t \varphi_\varepsilon$. Then, we integrate over Q_t , as usual. We obtain

$$\begin{aligned} & \frac{1}{2} \|\partial_t \varphi_\varepsilon(t)\|^2 + \int_0^t \|B^\sigma \partial_t \varphi_\varepsilon(s)\|^2 + \int_{Q_t} \beta'_\varepsilon(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2 \\ &= \frac{1}{2} \|\partial_t \varphi_\varepsilon(0)\|^2 + \int_{Q_t} (\ell \partial_t \vartheta_\varepsilon - \pi'(\varphi_\varepsilon) \partial_t \varphi_\varepsilon) \partial_t \varphi_\varepsilon. \end{aligned} \quad (5.8)$$

The last integral can be trivially estimated owing to the Lipschitz continuity of π , the Young inequality and (5.2)–(5.3) as follows

$$\int_{Q_t} (\ell \partial_t \vartheta_\varepsilon - \pi'(\varphi_\varepsilon) \partial_t \varphi_\varepsilon) \partial_t \varphi_\varepsilon \leq c \int_{Q_t} |\partial_t \varphi_\varepsilon|^2 + c \int_{Q_t} |\partial_t \vartheta_\varepsilon|^2 \leq c.$$

On the other hand, we have from (4.6)

$$\|\partial_t \varphi_\varepsilon(0)\| \leq \ell \|\vartheta_0\| + \|B^{2\sigma} \varphi_0\| + \|\beta_\varepsilon(\varphi_0)\| + \|\pi(\varphi_0)\| \leq \|\beta^\circ(\varphi_0)\| + c = c$$

since $|\beta_\varepsilon(r)| \leq |\beta^\circ(r)|$ for every $r \in D(\beta)$ (see, e.g., [10, Prop. 2.6, p. 28]). Hence, we conclude that

$$\|\partial_t \varphi_\varepsilon\|_{L^\infty(0, T; H) \cap L^2(0, T; V_B^\sigma)} \leq c. \quad (5.9)$$

\square

Proof of Theorem 2.10. We show how to modify the derivation of estimate (5.9) in the nonlinear case (2.19) by also assuming (2.39). The difference is the right-hand side of (5.8) one obtains by differentiating (4.6) with respect to time and then testing by $\partial_t \varphi_\varepsilon$. Namely, it contains the more complicated terms

$$\int_{Q_t} \partial_t \vartheta_\varepsilon \ell(\varphi_\varepsilon) \partial_t \varphi_\varepsilon + \int_{Q_t} \vartheta_\varepsilon \ell'(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2.$$

The first one is treated as before by using the boundedness of ℓ . About the second one, we recall that ℓ' is bounded by (2.19); then, owing to the Hölder inequality and (2.39), and accounting for (5.3) and the compactness inequality (2.40), we have that

$$\begin{aligned} & \int_{Q_t} \vartheta_\varepsilon \ell'(\varphi_\varepsilon) |\partial_t \varphi_\varepsilon|^2 \leq c \int_0^t \|\vartheta_\varepsilon(s)\|_p \|\partial_t \varphi_\varepsilon(s)\|_q^2 ds \leq c \int_0^t \|\vartheta_\varepsilon(s)\|_{A, r} \|\partial_t \varphi_\varepsilon(s)\|_q^2 ds \\ & \leq c \int_0^t \|\partial_t \varphi_\varepsilon(s)\|_q^2 ds \leq \frac{1}{2} \int_0^t \|B^\sigma \partial_t \varphi_\varepsilon(s)\|^2 ds + c \int_0^t \|\partial_t \varphi_\varepsilon(s)\|^2 ds. \end{aligned}$$

By combining this with the modified (5.8) and applying the Gronwall lemma, we obtain (5.9). Thus, the proof is complete.

6 Longtime behavior

In this section, we prove Theorem 2.14. Thus, in addition to the assumption (2.15)–(2.18) on the structure and (2.21) on the initial data, we suppose that the other hypotheses of the statement are in force and study the ω -limit of the unique global solution. In order to show that ω is nonempty and for a further use, we need some global estimates on (ϑ, φ) on the half line $[0, +\infty)$. In developing our argument, we start from the approximating problem once more, which clearly has a unique solution $(\vartheta_\varepsilon, \varphi_\varepsilon)$ defined in the whole of $[0, +\infty)$. We notice that the convergence of the approximating solution to (ϑ, φ) we have proved in Section 5 holds for every T (and for the whole family, i.e., not only for a subsequence), thank to our uniqueness result proved in Section 3.

First global estimate. As we did to prove (5.2), we test (4.5) and (4.6) by ϑ_ε and $\partial_t \varphi_\varepsilon$, respectively, integrate over Q_t and sum up. However, in contrast to the previous argument, we avoid adding the same term to both sides of the equality we get. Since the terms involving ℓ cancel each other and (2.41) holds, we obtain

$$\begin{aligned} & \frac{1}{2} \|\vartheta_\varepsilon(t)\|^2 + \int_0^t \|A^r \vartheta_\varepsilon(s)\|^2 ds + \int_{Q_t} |\partial_t \varphi_\varepsilon|^2 + \frac{1}{2} \|B^\sigma \varphi_\varepsilon(t)\|^2 + \alpha \|\varphi_\varepsilon(t)\|^2 - C \\ & \leq \frac{1}{2} \|\vartheta_0\|^2 + \frac{1}{2} \|B^\sigma \varphi_0\|^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_0) + \int_{Q_t} f \vartheta_\varepsilon. \end{aligned}$$

The first three terms on the right-hand side are bounded uniformly with respect to ε by the first condition in (4.1) and (2.21). Concerning the last one, we owe to (2.43) and have that

$$\int_{Q_t} f \vartheta_\varepsilon \leq \sup_{0 \leq s \leq t} \|\vartheta_\varepsilon(s)\| \int_0^t \|f(s)\| ds \leq \frac{1}{4} \sup_{0 \leq s \leq t} \|\vartheta_\varepsilon(s)\|^2 + \|f\|_{L^1(0, +\infty; H)}^2.$$

At this point, it is straightforward to conclude that

$$\|\vartheta_\varepsilon\|_{L^\infty(0, +\infty; H)} + \|A^r \vartheta_\varepsilon\|_{L^2(0, +\infty; H)} + \|\varphi_\varepsilon\|_{L^\infty(0, +\infty; V_B^\sigma)} + \|\partial_t \varphi_\varepsilon\|_{L^2(0, +\infty; H)} \leq c. \quad (6.1)$$

Second global estimate. We test (4.5) by $\partial_t \vartheta_\varepsilon$. By integrating over Q_t and using the Young inequality, the boundedness of ℓ and assumption (2.43) on f , we obtain

$$\begin{aligned} & \int_{Q_t} |\partial_t \vartheta_\varepsilon|^2 + \frac{1}{2} \|A^r \vartheta_\varepsilon(t)\|^2 = \int_{Q_t} (f - \ell(\varphi_\varepsilon) \partial_t \varphi_\varepsilon) \partial_t \vartheta_\varepsilon \\ & \leq \frac{1}{2} \int_{Q_t} |\partial_t \vartheta_\varepsilon|^2 + c \|f\|_{L^2(0, +\infty; H)}^2 + c \int_{Q_t} |\partial_t \varphi_\varepsilon|^2. \end{aligned}$$

Hence, (5.2) immediately yields

$$\|\vartheta_\varepsilon\|_{L^\infty(0, +\infty; V_A^1)} + \|\partial_t \vartheta_\varepsilon\|_{L^2(0, +\infty; H)} \leq c. \quad (6.2)$$

Basic global estimates. By letting ε tend to zero in (6.1)–(6.2), we see that the solution (ϑ, φ) we are studying enjoys the properties

$$\vartheta \in L^\infty(0, +\infty; V_A^r) \quad \text{and} \quad \varphi \in L^\infty(0, +\infty; V_B^\sigma) \subset L^\infty(0, +\infty; H) \quad (6.3)$$

$$\int_0^{+\infty} \|A^r \vartheta(s)\|^2 ds + \int_0^{+\infty} \|\partial_t \vartheta(s)\|^2 ds + \int_0^{+\infty} \|\partial_t \varphi(s)\|^2 ds < +\infty. \quad (6.4)$$

In particular, the ω -limit ω is nonempty.

The next step consists in proving the properties of the elements of ω we have stated. Thus, we fix $(\vartheta_\omega, \varphi_\omega) \in \omega$ and a corresponding sequence $\{t_n\}$ as in the definition (2.44), and, for every n , we study the limits on a fixed time interval $(0, T)$ of the functions ϑ^n and φ^n defined by

$$\vartheta^n(t) := \vartheta(t + t_n) \quad \text{and} \quad \varphi^n(t) := \varphi(t + t_n) \quad \text{for } t \in [0, T]. \quad (6.5)$$

The global estimates (6.3)–(6.4) immediately yield that

$$\begin{aligned} & \|\vartheta^n\|_{L^\infty(0, T; H)} + \|\varphi^n\|_{L^\infty(0, T; V_B^\sigma)} \leq c \\ & \lim_{n \rightarrow \infty} \left(\int_0^T \|A^r \vartheta^n(s)\|^2 ds + \int_0^T \|\partial_t \vartheta^n(s)\|^2 ds + \int_0^T \|\partial_t \varphi^n(s)\|^2 ds \right) = 0. \end{aligned}$$

By standard compactness results it follows that

$$\vartheta^n \rightarrow \vartheta^\infty \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V_A^r) \quad (6.6)$$

$$\varphi^n \rightarrow \varphi^\infty \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V_B^\sigma) \quad (6.7)$$

at least for a subsequence, where ϑ^∞ and φ^∞ satisfy

$$A^r \vartheta^\infty = 0, \quad \partial_t \vartheta^\infty = 0 \quad \text{and} \quad \partial_t \varphi^\infty = 0 \quad \text{a.e. in } Q. \quad (6.8)$$

In particular, both ϑ^∞ and φ^∞ are time-independent, so that we can define the elements $\vartheta_s \in V_A^r$ and $\varphi_s \in V_B^\sigma$ by setting

$$\vartheta_s := \vartheta^\infty(t) \quad \text{and} \quad \varphi_s := \varphi^\infty(t) \quad \text{for every } t \in [0, T].$$

Our aim is to show that $(\vartheta_\omega, \varphi_\omega) = (\vartheta_s, \varphi_s)$ and that (ϑ_s, φ_s) solves (2.46)–(2.47). By the weak convergence in $C^0([0, T]; H)$ implied by the weak convergence in $H^1(0, T; H)$, we have that

$$\vartheta_s = \vartheta^\infty(0) = \lim_{n \rightarrow \infty} \vartheta^n(0) = \lim_{n \rightarrow \infty} \vartheta(t_n) = \vartheta_\omega$$

where the limits are understood in the weak topology of H . Similarly, we obtain that $\varphi_s = \varphi_\omega$. As far as the limiting system is concerned, equation (2.46) follows from the first equality in (6.8). It remains to prove (2.47). To this end, we observe that the pair (ϑ^n, φ^n) obviously satisfies (2.27), whence also (2.29). On the other hand, we can invoke the compact embedding $V_B^\sigma \subset H$ (see (2.14)) and apply, e.g., [35, Sect. 8, Cor. 4] to obtain

$$\varphi^n \rightarrow \varphi^\infty \quad \text{strongly in } C^0([0, T]; H). \quad (6.9)$$

By Lipschitz continuity, it follows that

$$\pi(\varphi^n) \rightarrow \pi(\varphi^\infty) \quad \text{strongly in } C^0([0, T]; H). \quad (6.10)$$

Now, we prove that

$$\vartheta^n \ell(\varphi^n) \rightarrow \vartheta^\infty \ell(\varphi^\infty) \quad \text{weakly star in } L^\infty(0, T; H) \quad (6.11)$$

by distinguishing the cases of the statement that concern ℓ . If ℓ is a constant, then (6.11) trivially follows from (6.6). In the opposite case, we invoke (2.42) and apply [35, Sect. 8, Cor. 4]. Hence, from (6.6)–(6.7) we deduce that

$$\begin{aligned} \vartheta^n &\rightarrow \vartheta^\infty \quad \text{weakly star in } L^\infty(0, T; L^p(\Omega)), \\ \varphi^n &\rightarrow \varphi^\infty \quad \text{strongly in } C^0([0, T]; L^q(\Omega)), \quad \text{whence} \\ \ell(\varphi^n) &\rightarrow \ell(\varphi^\infty) \quad \text{strongly in } C^0([0, T]; L^q(\Omega)). \end{aligned}$$

Thus, (6.11) follows also in this case since $(1/p) + (1/q) = 1/2$. We remark that (6.10), (6.11) and (6.9) imply (at least)

$$\pi(\varphi^n) \varphi^n \rightarrow \varphi(\varphi^\infty) \varphi^\infty \quad \text{and} \quad \vartheta^n \ell(\varphi^n) \varphi^n \rightarrow \vartheta^\infty \ell(\varphi^\infty) \varphi^\infty \quad \text{weakly in } L^1(Q).$$

At this point, we can easily let n tend to infinity in (2.29) written for (ϑ^n, φ^n) . By also accounting for the lower semicontinuity of the convex function $v \mapsto \int_Q \widehat{\beta}(v)$ in $L^2(Q)$, we have that

$$\begin{aligned} &\int_Q \widehat{\beta}(\varphi^\infty) + \int_0^T (B^\sigma \varphi^\infty(t), B^\sigma(\varphi^\infty(t) - v(t))) dt \\ &\leq \liminf_{\varepsilon \searrow 0} \int_Q \widehat{\beta}(\varphi^\varepsilon) + \liminf_{n \rightarrow \infty} \int_0^T (B^\sigma \varphi^n(t), B^\sigma(\varphi^n(t) - v(t))) dt \\ &\leq \liminf_{n \rightarrow \infty} \left(\int_Q \widehat{\beta}(\varphi^n) + \int_0^T (B^\sigma \varphi^n(t), B^\sigma(\varphi^n(t) - v(t))) dt \right) \\ &\leq \liminf_{n \rightarrow \infty} \left(\int_Q (\vartheta^n \ell(\varphi^n) - \partial_t \varphi^n - \pi(\varphi^n))(\varphi^n - v) + \int_Q \widehat{\beta}_\varepsilon(v) \right) \\ &= \int_Q (\vartheta^\infty \ell(\varphi^\infty) - \partial_t \varphi^\infty - \pi(\varphi^\infty))(\varphi^\infty - v) + \int_Q \widehat{\beta}(v) \end{aligned}$$

where we have kept $\partial_t \varphi^\infty$ for clarity, even though it vanishes. Thus, (2.29) holds true for $(\vartheta^\infty, \varphi^\infty)$. This implies that (2.27) holds as well. But the latter coincides with (2.47) and the proof is complete.

7 Convergence to a phase relaxation problem

In this section we discuss the asymptotic behavior of the solution to our problem as $\sigma \searrow 0$.

We assume that (2.15)–(2.18) are satisfied,

$$\ell \text{ is a constant,} \quad (7.1)$$

$$\pi(v) = -\gamma v \quad \text{for all } v \in \mathbb{R}, \text{ with a fixed constant } \gamma \geq 0, \quad (7.2)$$

and there exist some $\sigma_0 > 0$ and some family of data $\{f_\sigma, \vartheta_{0,\sigma}, \varphi_{0,\sigma}\}$ such that

$$f_\sigma \rightarrow f \quad \text{in } L^2(0, T; H) \quad \text{as } \sigma \searrow 0, \quad (7.3)$$

$$\vartheta_{0,\sigma} \rightarrow \vartheta_0 \quad \text{in } V_A^r \quad \text{as } \sigma \searrow 0, \quad (7.4)$$

$$\begin{aligned} \varphi_{0,\sigma} \in V_B^\sigma \quad \text{and} \quad \|\varphi_{0,\sigma}\|_{V_B^\sigma} + \|\widehat{\beta}(\varphi_{0,\sigma})\|_{L^1(\Omega)} \leq c \quad \text{for all } \sigma \in (0, \sigma_0], \\ \varphi_{0,\sigma} \rightarrow \varphi_0 \quad \text{in } H \quad \text{as } \sigma \searrow 0. \end{aligned} \quad (7.5)$$

Concerning (7.5), we just note that if $\varphi_0 \in V_B^{\sigma_0}$ with $\widehat{\beta}(\varphi_0) \in L^1(\Omega)$ (cf. (2.21)), then the constant sequence $\varphi_{0,\sigma} = \varphi_0$ directly works in (7.5).

We are dealing with the solution $(\vartheta_\sigma, \varphi_\sigma)$ to the system (cf. (2.26)–(2.28))

$$\partial_t \vartheta_\sigma + \ell \partial_t \varphi_\sigma + A^{2r} \vartheta_\sigma = f_\sigma \quad \text{a.e. in } Q, \quad (7.6)$$

$$\begin{aligned} (\partial_t \varphi_\sigma(t), \varphi_\sigma(t) - v) + (B^\sigma \varphi_\sigma(t), B^\sigma(\varphi_\sigma(t) - v)) + \int_\Omega \widehat{\beta}(\varphi_\sigma(t)) - (\gamma \varphi_\sigma(t), \varphi_\sigma(t) - v) \\ \leq (\ell \vartheta_\sigma(t), \varphi_\sigma(t) - v) + \int_\Omega \widehat{\beta}(v) \quad \text{for a.a. } t \in (0, T) \text{ and every } v \in V_B^\sigma, \end{aligned} \quad (7.7)$$

$$\vartheta_\sigma(0) = \vartheta_{0,\sigma} \quad \text{and} \quad \varphi_\sigma(0) = \varphi_{0,\sigma}, \quad (7.8)$$

where $(\vartheta_\sigma, \varphi_\sigma)$ satisfies (cf. (2.23)–(2.25))

$$\vartheta_\sigma \in H^1(0, T; H) \cap L^\infty(0, T; V_A^r) \cap L^2(0, T; V_A^{2r}), \quad (7.9)$$

$$\varphi_\sigma \in H^1(0, T; H) \cap L^\infty(0, T; V_B^\sigma), \quad (7.10)$$

$$\widehat{\beta}(\varphi_\sigma) \in L^1(Q). \quad (7.11)$$

The convergence theorem we prove is as follows.

Theorem 7.1. *Under the assumptions (2.15)–(2.18) and (7.1)–(7.5), the family of solutions $(\vartheta_\sigma, \varphi_\sigma)$ to the problem (7.6)–(7.11) satisfies*

$$\begin{aligned} \vartheta_\sigma \rightarrow \vartheta \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V_A^r) \cap L^2(0, T; V_A^{2r}) \\ \text{and strongly in } C^0([0, T]; H), \end{aligned} \quad (7.12)$$

$$\varphi_\sigma \rightarrow \varphi \quad \text{weakly in } H^1(0, T; H) \quad (7.13)$$

as $\sigma \searrow 0$, where the limit pair (ϑ, φ) is the unique solution to the problem

$$\partial_t \vartheta + \ell \partial_t \varphi + A^{2r} \vartheta = f \quad \text{a.e. in } Q, \quad (7.14)$$

$$\begin{aligned} \partial_t \varphi + \varphi - P\varphi + \xi - \gamma \varphi = \ell \vartheta, \quad \text{for some } \xi \in L^2(0, T; H) \text{ satisfying} \\ \xi \in \beta(\varphi) \quad \text{a.e. in } Q, \end{aligned} \quad (7.15)$$

$$\vartheta(0) = \vartheta_0, \quad \varphi(0) = \varphi_0 \quad (7.16)$$

and, in (7.15), P denotes the H -projection operator on the kernel of the operator B .

Remark 7.2. With reference to Remark 2.1, let us point out that in the case whether B is the Laplace operator $-\Delta$ with Neumann boundary conditions, the operator P maps any element $v \in H$ into a constant function, which is proportional to the mean value of v as the first eigenfunction of B is $\eta_1 = |\Omega|^{-1/2}$.

Remark 7.3. It turns out that Theorem 7.1 works under the special assumption (7.2) of a linear function π that is the derivative of a concave quadratic function $\widehat{\pi}$. This is of course a special situation, but let us point out that all the three significant examples of potentials (1.3)–(1.5) contemplate exactly a *quadratic concave perturbation*, along with a convex and possibly singular or nonsmooth function.

The whole section is devoted to the proof of this theorem. About the uniqueness of the limit (ϑ, φ) and the related continuous dependence property with respect to the data $(f, \vartheta_0, \varphi_0)$, it is not difficult to somehow reproduce the proof given in Section 3 for the case of a constant ℓ and derive an estimate similar to (3.4), which leads to

Proposition 7.4. *Assume that (2.15)–(2.18) and (7.1)–(7.2) are satisfied. Moreover, let the data f , ϑ_0 and φ_0 satisfy (7.3)–(7.5) for some family $\{f_\sigma, \vartheta_{0,\sigma}, \varphi_{0,\sigma}\}$. Then, there exists a unique solution*

$$(\vartheta, \varphi) \in (H^1(0, T; H) \cap L^\infty(0, T; V_A^r) \cap L^2(0, T; V_A^{2r})) \times H^1(0, T; H)$$

to the problem (7.14)–(7.16). Moreover, if $(f_i, \vartheta_{0i}, \varphi_{0i})$, $i = 1, 2$, are two choices of the data and (ϑ_i, φ_i) are the corresponding solutions, then we have

$$\begin{aligned} & \|\vartheta_1 - \vartheta_2\|_{L^2(0, T; H)} + \|1 * (\vartheta_1 - \vartheta_2)\|_{L^\infty(0, T; V_A^r)} \\ & + \|\varphi_1 - \varphi_2\|_{L^\infty(0, T; H)} + \|\varphi_1 - P\varphi_1 - \varphi_2 + P\varphi_2\|_{L^2(0, T; H)} \\ & \leq K(\|1 * (f_1 - f_2)\|_{L^2(0, T; H)} + \|\vartheta_{01} - \vartheta_{02}\| + \|\varphi_{01} - \varphi_{02}\|) \end{aligned} \quad (7.17)$$

for some constant K depending only on ℓ, γ, T .

Now, we concentrate our efforts on the proof of the remaining convergence properties stated in Theorem 7.1. We start by proving the following auxiliary result.

Lemma 7.5. *Assume that $v \in V_B^{\sigma_0}$ for some $\sigma_0 > 0$. Then we have that $B^\sigma v$ is well defined for all $\sigma \in [0, \sigma_0]$ and*

$$B^\sigma v \rightarrow v - Pv \quad \text{strongly in } H \quad \text{as } \sigma \searrow 0, \quad (7.18)$$

where, as above, P denotes the H -projection on $\{v \in D(B) : Bv = 0\}$.

Proof. The first part of the statement follows easily from (2.7) and (2.8). In particular, we note that

$$B^\sigma v = \sum_{j=1}^{\infty} \mu_j^\sigma (v, \eta_j) \eta_j \quad \text{for all } 0 < \sigma \leq \sigma_0, \quad v - Pv = \sum_{\mu_j > 0} (v, \eta_j) \eta_j.$$

Then, we have to prove that

$$B^\sigma v - (v - Pv) = \sum_{\mu_j > 0} (\mu_j^\sigma - 1) (v, \eta_j) \eta_j \rightarrow 0 \quad \text{strongly in } H \quad \text{as } \sigma \searrow 0.$$

In view of (2.3)–(2.5), it is sufficient to verify that

$$\|B^\sigma v - (v - Pv)\|^2 = \sum_{\mu_j > 0} (\mu_j^\sigma - 1)^2 |(v, \eta_j)|^2$$

tends to 0 as $\sigma \searrow 0$. We observe that $(\mu_j^\sigma - 1)^2 \leq 1$ if $\mu_j \leq 1$ and $(\mu_j^\sigma - 1)^2 \leq \mu_j^{2\sigma_0}$ if $\mu_j > 1$. Hence, we have that

$$\sum_{\mu_j > 0} (\mu_j^\sigma - 1)^2 |(v, \eta_j)|^2 \leq \sum_{j=1}^{\infty} (1 + \mu_j^{2\sigma_0}) |(v, \eta_j)|^2 = \|v\|^2 + \|B^{\sigma_0} v\|^2 < +\infty.$$

Therefore, the reader can realize that it is possible to apply the Lebesgue dominated convergence theorem, with respect to the counting measure $\#$, to the family of functions

$$f_\sigma(j) = \begin{cases} 0 & \text{if } \mu_j = 0 \\ (\mu_j^\sigma - 1)^2 |(v, \eta_j)|^2 & \text{if } \mu_j > 0 \end{cases}, \quad j \in \mathbb{N}.$$

Since

$$\begin{aligned} f_\sigma &\rightarrow 0 \quad \text{pointwise in } \mathbb{N} \quad \text{as } \sigma \searrow 0, \\ 0 \leq f_\sigma(j) &\leq g(j) := (1 + \mu_j^{2\sigma_0}) |(v, \eta_j)|^2 \quad \text{for all } j \in \mathbb{N}, \end{aligned}$$

and g is summable with respect to $\#$ by the above calculation, we can conclude that

$$\|B^\sigma v - (v - Pv)\|^2 = \int_{\mathbb{N}} f_\sigma(j) d\#(j) \rightarrow 0$$

and (7.18) and the lemma are completely proved. \square

Next, we recall and take advantage of the uniform estimates pointed out in Section 5. Arguing as in the derivation of (5.2), recalling (2.10) and using the lower semicontinuity properties when passing to the limit as $\varepsilon \searrow 0$, from (5.1) we obtain

$$\begin{aligned} &\frac{1}{2} \|\vartheta_\sigma(t)\|^2 + \int_0^t \|A^r \vartheta_\sigma(s)\|^2 ds + \frac{1}{2} \int_{Q_t} |\partial_t \varphi_\sigma|^2 \\ &+ \frac{1}{2} (\|\varphi_\sigma(t)\|^2 + \|B^\sigma \varphi_\sigma(t)\|^2) + \int_\Omega \widehat{\beta}(\varphi_\sigma(t)) \\ &\leq c + \frac{1}{2} \|\vartheta_{0,\sigma}\|^2 + \frac{1}{2} (\|\varphi_{0,\sigma}\|^2 + \|B^\sigma \varphi_{0,\sigma}\|^2) \\ &+ \int_\Omega \widehat{\beta}(\varphi_{0,\sigma}) + \frac{1}{2} \int_0^t \|f_\sigma(s) + (1 + \gamma)\varphi_\sigma(s)\|^2 ds. \end{aligned}$$

Hence, by virtue of (7.3)–(7.5) and applying the Gronwall lemma, we deduce that

$$\begin{aligned} &\|\vartheta_\sigma\|_{L^\infty(0,T;H) \cap L^2(0,T;V_\lambda^r)} + \|\varphi_\sigma\|_{H^1(0,T;H)} \\ &+ \|B^\sigma \varphi_\sigma\|_{L^\infty(0,T;H)} + \|\widehat{\beta}(\varphi_\sigma)\|_{L^\infty(0,T;L^1(\Omega))} \leq c. \end{aligned} \quad (7.19)$$

Now, we can test (7.6) by $\partial_t \vartheta_\sigma$ and integrate with respect to time obtaining

$$\int_{Q_t} |\partial_t \vartheta_\sigma|^2 + \frac{1}{2} \|A^r \vartheta_\sigma(t)\|^2 = \frac{1}{2} \|A^r \vartheta_{0,\sigma}\|^2 + \int_{Q_t} (f_\sigma - \ell \partial_t \varphi_\sigma) \partial_t \vartheta_\sigma.$$

Then, the Young inequality, (7.19) and (7.3)–(7.4) enable us to infer that

$$\|\vartheta_\sigma\|_{H^1(0,T;H) \cap L^\infty(0,T;V_\lambda^r)} \leq c. \quad (7.20)$$

Moreover, in view of (7.3) and by a comparison in (7.6) we have that

$$\|\vartheta_\sigma\|_{L^2(0,T;V_A^{2r})} \leq c. \quad (7.21)$$

Due to (7.19)–(7.21), we can pass to the limit as $\sigma \searrow 0$, at the beginning for a subsequence, by using standard weak and known strong compactness results (see, e.g., [35, Sect. 8, Cor. 4]). Hence, we find out the convergence (7.12)–(7.13) to ϑ and φ , along with

$$B^\sigma \varphi_\sigma \rightarrow \zeta \quad \text{weakly star in } L^\infty(0, T; H). \quad (7.22)$$

In a first verification, inspired by Lemma 7.5, we aim to check the weak star limit ζ in (7.22) is nothing but $\varphi - P\varphi$. Actually, we show that

$$B^\sigma \varphi_\sigma \rightarrow \varphi - P\varphi \quad \text{weakly in } L^2(0, T; H), \quad (7.23)$$

by verifying this property with respect to a dense subset of $L^2(0, T; H)$. Indeed, thanks to (2.5) it suffices to prove that

$$\begin{aligned} \int_0^T (B^\sigma \varphi_\sigma(t), \psi(t)\eta_j) dt &\rightarrow \int_0^T (\varphi(t) - P\varphi(t), \psi(t)\eta_j) dt \\ &\text{for all } \psi \in L^2(0, T) \text{ and } j \in \mathbb{N}. \end{aligned} \quad (7.24)$$

Note that the integrals in (7.24) all vanish if the index j is such that the eigenvalue μ_j is equal to 0. If instead $\mu_j > 0$, then with the help of (2.8) and (7.13) we have that

$$\int_0^T (B^\sigma \varphi_\sigma(t), \psi(t)\eta_j) dt = \mu_j^\sigma \int_0^T (\varphi_\sigma(t), \psi(t)\eta_j) dt \rightarrow \int_0^T (\varphi(t), \psi(t)\eta_j) dt$$

as $\sigma \searrow 0$. Moreover, η_j is orthogonal to the kernel of B . All this means that in both cases there is convergence to $\int_0^T (\varphi(t) - P\varphi(t), \psi(t)\eta_j) dt$ and (7.24) is ensured.

At this point, it remains to prove that the limiting pair (ϑ, φ) solves the system (7.14)–(7.16). The initial conditions (7.16) hold true by virtue of (7.12)–(7.13), (7.8) and (7.4)–(7.5). In particular, note that (7.13) guarantees at least that

$$\varphi_\sigma(t) \rightarrow \varphi(t) \quad \text{weakly in } H, \text{ for all } t \in [0, T]. \quad (7.25)$$

On the other hand, recalling (7.3) and passing to the limit in (7.6) we arrive at (7.14). Next, we let the test function v in (7.7) to depend also on time, taking $v \in L^2(0, T; V_B^{\sigma_0})$, and multiply the inequality by $e^{-2\gamma t}$, then integrating over $(0, T)$. We obtain

$$\begin{aligned} &\int_0^T (e^{-\gamma t}(\partial_t \varphi_\sigma(t) - \gamma \varphi_\sigma(t)), e^{-\gamma t}(\varphi_\sigma - v)(t)) dt \\ &+ \int_0^T e^{-2\gamma t} (B^\sigma \varphi_\sigma(t), B^\sigma(\varphi_\sigma - v)(t)) dt + \int_0^T \int_\Omega e^{-2\gamma t} \widehat{\beta}(\varphi_\sigma(t)) dt \\ &\leq \int_0^T e^{-2\gamma t} (\ell \vartheta_\sigma(t), (\varphi_\sigma - v)(t)) dt + \int_0^T \int_\Omega e^{-2\gamma t} \widehat{\beta}(v(t)) dt \\ &\text{for every } v \in L^2(0, T; V_B^{\sigma_0}). \end{aligned} \quad (7.26)$$

Let us introduce the family of functions

$$\rho_\sigma(t) := e^{-\gamma t} \varphi_\sigma(t), \quad t \in [0, T],$$

and point out that (cf. (7.13) and (7.25))

$$\rho_\sigma \rightarrow \rho \quad \text{weakly in } H^1(0, T; H), \quad \text{where } \rho(t) := e^{-\gamma t} \varphi(t), \quad t \in [0, T], \quad (7.27)$$

$$\rho_\sigma(t) \rightarrow \rho(t) \quad \text{weakly in } H, \quad \text{for all } t \in [0, T]. \quad (7.28)$$

Then, we observe that

$$\begin{aligned} & \int_0^T (e^{-\gamma t} (\partial_t \varphi_\sigma(t) - \gamma \varphi_\sigma(t)), e^{-\gamma t} (\varphi_\sigma - v)(t)) dt \\ &= \int_0^T (\partial_t \rho_\sigma(t), \rho_\sigma(t) - e^{-\gamma t} v(t)) dt \\ &= \frac{1}{2} \|\rho_\sigma(T)\|^2 - \frac{1}{2} \|\varphi_{0,\sigma}\|^2 - \int_0^T (\partial_t \rho_\sigma(t), e^{-\gamma t} v(t)) dt \end{aligned}$$

and, on account of (7.5), (7.27), (7.28) and the weak lower semicontinuity of norms, we have that

$$\begin{aligned} & \int_0^T (e^{-\gamma t} (\partial_t \varphi(t) - \gamma \varphi(t)), e^{-\gamma t} (\varphi - v)(t)) dt \\ &= \int_0^T (\partial_t \rho(t), \rho(t) - e^{-\gamma t} v(t)) dt \\ &= \frac{1}{2} \|\rho(T)\|^2 - \frac{1}{2} \|\varphi_0\|^2 - \int_0^T (\partial_t \rho(t), e^{-\gamma t} v(t)) dt \\ &\leq \liminf_{\sigma \searrow 0} \left\{ \frac{1}{2} \|\rho_\sigma(T)\|^2 - \frac{1}{2} \|\varphi_{0,\sigma}\|^2 - \int_0^T (\partial_t \rho_\sigma(t), e^{-\gamma t} v(t)) dt \right\} \\ &= \liminf_{\sigma \searrow 0} \int_0^T (e^{-\gamma t} (\partial_t \varphi_\sigma(t) - \gamma \varphi_\sigma(t)), e^{-\gamma t} (\varphi_\sigma - v)(t)) dt. \end{aligned} \quad (7.29)$$

Similarly, we recall (7.23) and point out that $B^\sigma v \rightarrow v - Pv$ strongly in $L^2(0, T; H)$: indeed, this is a consequence of Lemma 7.5, the bounds

$$\|B^\sigma v(t)\| \leq \|v(t)\|_{B,\sigma} \leq c \|v(t)\|_{B,\sigma_0} \quad \text{for a.a. } t \in (0, T) \text{ and every } \sigma \in (0, \sigma_0],$$

$$\text{along with } \int_0^T \|v(t)\|_{B,\sigma_0}^2 dt = \|v\|_{L^2(0,T;V_B^{\sigma_0})}^2 < +\infty,$$

and the Lebesgue dominated convergence theorem. Hence, as $\varphi - P\varphi$ is orthogonal to every element of the kernel of B , we infer that

$$\begin{aligned} & \int_0^T e^{-2\gamma t} ((\varphi - P\varphi)(t), (\varphi - v)(t)) dt \\ &= \int_0^T e^{-2\gamma t} ((\varphi - P\varphi)(t), (\varphi - P\varphi)(t) - (v - Pv)(t)) dt \\ &\leq \liminf_{\sigma \searrow 0} \int_0^T e^{-2\gamma t} (B^\sigma \varphi_\sigma(t), B^\sigma (\varphi_\sigma - v)(t)) dt. \end{aligned} \quad (7.30)$$

Next, we observe that the function

$$v \mapsto \int_0^T \int_{\Omega} e^{-2\gamma t} \widehat{\beta}(v(t)) dt$$

is convex and lower semicontinuous in $L^2(0, T; H)$, as one can easily verify. Therefore, since φ_{σ} weakly converges to φ in $L^2(0, T; H)$ (see (7.13)), we have that

$$\int_0^T \int_{\Omega} e^{-2\gamma t} \widehat{\beta}(\varphi) dt \leq \liminf_{\sigma \searrow 0} \int_0^T \int_{\Omega} e^{-2\gamma t} \widehat{\beta}(\varphi_{\sigma}(t)) dt. \quad (7.31)$$

Now, we take advantage of (7.29)–(7.31) and, in view of (7.12) and (7.13), from (7.26) we deduce that

$$\begin{aligned} & \int_0^T (e^{-\gamma t} (\partial_t \varphi(t) - \gamma \varphi(t)), e^{-\gamma t} (\varphi - v)(t)) dt \\ & + \int_0^T e^{-2\gamma t} ((\varphi - P\varphi)(t), (\varphi - v)(t)) dt + \int_0^T \int_{\Omega} e^{-2\gamma t} \widehat{\beta}(\varphi) dt \\ & \leq \liminf_{\sigma \searrow 0} \left(\int_0^T (e^{-\gamma t} (\partial_t \varphi_{\sigma}(t) - \gamma \varphi_{\sigma}(t)), e^{-\gamma t} (\varphi_{\sigma} - v)(t)) dt \right. \\ & \quad \left. + \int_0^T e^{-2\gamma t} (B^{\sigma} \varphi_{\sigma}(t), B^{\sigma} (\varphi_{\sigma} - v)(t)) dt + \int_0^T \int_{\Omega} e^{-2\gamma t} \widehat{\beta}(\varphi_{\sigma}(t)) dt \right) \\ & \leq \int_0^T e^{-2\gamma t} (\ell \vartheta(t), (\varphi - v)(t)) dt + \int_0^T \int_{\Omega} e^{-2\gamma t} \widehat{\beta}(v(t)) dt \\ & \quad \text{for every } v \in L^2(0, T; V_B^{\sigma_0}). \end{aligned}$$

Therefore, at the end we derive the same inequality as (7.26) for the limit functions ϑ and φ . Moreover, it is not difficult to check that this inequality is equivalent to

$$\begin{aligned} & (\partial_t \varphi(t), \varphi(t) - v) + (\varphi(t) - P\varphi(t), \varphi(t) - v) + \int_{\Omega} \widehat{\beta}(\varphi(t)) - (\gamma \varphi(t), \varphi(t) - v) \\ & \leq (\ell \vartheta(t), \varphi(t) - v) + \int_{\Omega} \widehat{\beta}(v) \quad \text{for a.a. } t \in (0, T) \text{ and every } v \in V_B^{\sigma_0}. \end{aligned} \quad (7.32)$$

Then, by a density argument it is straightforward to infer that (7.32) holds true for all $v \in H$, whence the definition of subdifferential for the convex functional

$$v \in H \mapsto \int_{\Omega} \widehat{\beta}(v) \in [0, +\infty]$$

enables us to conclude that for a.a. $t \in (0, T)$

$$\xi(t) := (\ell \vartheta - \partial_t \varphi - \varphi + P\varphi + \gamma \varphi)(t) \in \beta(\varphi(t)) \quad \text{a.e. in } \Omega. \quad (7.33)$$

It is easy now to see that (7.33) finally leads to (7.15).

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