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Abstract
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Closure spaces, Spatial logics

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On the Expressive Power of IMLC and ISLCS

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1 Introduction

In these notes we prove that, in quasi-discrete closure models, the ISLCS forward (backword) conditional reachability operator can be expressed using a (possibly) infinite disjunction of nested formulas using only conjunction and the IMLC backward (forward) proximity operator.

2 Preliminaries and Terminology

We recall the notion of path as a continuous function; for all other definitions and notation we refer to [2, 1].

Definition 1 (Continuous function). *Function $f : X_1 \to X_2$ is a* continuous *function from $(X_1, \mathcal{C}_1)$ to $(X_2, \mathcal{C}_2)$ if and only if for all sets $A \subseteq X_1$ we have $f(\mathcal{C}_1(A)) \subseteq \mathcal{C}_2(f(A))$.* •

Definition 2 (Index space). *An* index space *is a connected*[3] *CS $(I, \mathcal{C})$ equipped with a total order $\leqslant \subseteq I \times I$ with a bottom element 0. We often write $\iota_1 < \iota_2$ whenever $\iota_1 \leqslant \iota_2$ and $\iota_1 \neq \iota_2$, (ι_1, ι_2) for $\{\iota \mid \iota_1 < \iota < \iota_2\}$, $[\iota_1, \iota_2)$ for $\{\iota \mid \iota_1 \leq \iota < \iota_2\}$, and $(\iota_1, \iota_2]$ for $\{\iota \mid \iota_1 < \iota \leq \iota_2\}$.* •

For QdCMs, index spaces are intervals $[0, n]$ over the set of natural numbers, with the successor relation and the closure operator $\mathcal{C}_{\mathtt{succ}}$ induced by such a relation.

Definition 3 (Path). *A* path *in CS $(X, \mathcal{C})$ is a continuous function from an index space $\mathcal{J} = (I, \mathcal{C}^{\mathcal{J}})$ to $(X, \mathcal{C})$. A path π is* bounded *if there exists $\ell \in I$ such that $\pi(\iota) = \pi(\ell)$ for all ι such that $\ell \leqslant \iota$; we call the minimal such ℓ the* length *of π, written $\mathtt{len}(\pi)$.* •

We need some additional notation and terminology:

[3] Given CS $(X, \mathcal{C})$, $A \subseteq X$ is *connected* if it is *not* the union of two non-empty separated sets. Two subsets $A_1, A_2 \subseteq X$ are called *separated* if $A_1 \cap \mathcal{C}(A_2) = \emptyset = \mathcal{C}(A_1) \cap A_2$. CS $(X, \mathcal{C})$ is *connected* if X is connected.

Definition 4. *Given* IMLC *formulas* Φ_1 *and* Φ_2*, we let* $\Psi_1, \Psi_2 \ldots$ *be the following formulas:*

$$\Psi_1 = \overleftarrow{\mathcal{N}}\ \Phi_1$$
$$\Psi_2 = \overleftarrow{\mathcal{N}}\ (\Phi_2 \wedge \overleftarrow{\mathcal{N}}\ \Phi_1) = \overleftarrow{\mathcal{N}}\ (\Phi_2\ \wedge \Psi_1)$$
$$\Psi_3 = \overleftarrow{\mathcal{N}}\ (\Phi_2\ \wedge \Psi_2)$$
$$\vdots$$

so that, for $j \geq 1$ *we have:*

$$\Psi_{j+1} = \overleftarrow{\mathcal{N}}\ (\Phi_2\ \wedge \Psi_j) = \overbrace{\overleftarrow{\mathcal{N}}\ (\Phi_2 \wedge \overleftarrow{\mathcal{N}}\ (\ldots \overleftarrow{\mathcal{N}}\ (\Phi_2 \wedge \overleftarrow{\mathcal{N}}\ \Phi_1)))}^{j+1 \text{ nested } \overleftarrow{\mathcal{N}}}.$$

Definition 5. *We say that a* bounded *path* π *is* internal loops free *(ILF, or* canonical*, in the sequel) if and only if whenever* $\pi(i) = \pi(j)$ *for some* $i < j$ *then* $\pi(k) = \pi(i)$ *for all* $k \geq i$*.*

Intuitively, canonical paths are obtained by removing "redundant" cycles leaving only the last one, starting at the index that determines the length of the path.

Definition 6. *A path* π *is a* $\Phi_1[\Phi_2]$*-path if and only if there exists* $\ell \geq 0$ *such that* $\mathcal{M}, \pi(\ell) \models \Phi_1$ *and* $\mathcal{M}, \pi(j) \models \Phi_2$ *for all* $j \in (0, \ell)$*.*

We note that for each $\Phi_1[\Phi_2]$-path π there is a (shorter) canonical path π' such that $\pi(0) = \pi'(0)$ and π' is a $\Phi_1[\Phi_2]$-path as well.

3 The main result

Theorem 1. *For all* $x \in X$ *and* Ψ_j *as in Definition 4 the following holds:* $\mathcal{M}, x \models \overrightarrow{\rho}\ \Phi_1[\Phi_2]$ *if and only if* $\mathcal{M}, x \models \bigvee_{j \geq 1} \Psi_j$*.*

Proof. Suppose $\mathcal{M}, x \models \overrightarrow{\rho}\ \Phi_1[\Phi_2]$. This means that $\mathcal{M}, x \models \Phi_1$ or that there exists a $\Phi_1[\Phi_2]$-path rooted in x. In the first case, by definition, we have $x \in [\![\Phi_1]\!]$ and, since $A \subseteq \overleftarrow{\mathcal{C}}\ (A)$ for all $A \subseteq X$, we also have $x \in \overleftarrow{\mathcal{C}}\ ([\![\Phi_1]\!])$. So, by definition of $\overleftarrow{\mathcal{N}}$, we have $\mathcal{M}, x \models \overleftarrow{\mathcal{N}}\ \Phi_1$. Consequently, by definition of Ψ_1, $\mathcal{M}, x \models \Psi_1$, and thus $\mathcal{M}, x \models \bigvee_{j \geq 1} \Psi_j$. In the second case, there exists also a canonical $\Phi_1[\Phi_2]$-path π such that $\pi(0) = x$ and $\texttt{len}\,\pi = n$ for some $n \geq 1$; by Lemma 1 below, we get $\mathcal{M}, x \models \Psi_n$ for some $n \geq 1$, and thus $\mathcal{M}, x \models \bigvee_{j \geq 1} \Psi_j$.
Assume now $\mathcal{M}, x \models \bigvee_{j \geq 1} \Psi_j$. This means there exists $n \geq 1$ such that $\mathcal{M}, x \models \Psi_n$. By Lemma 2 below, there exists a $\Phi_1[\Phi_2]$-path π with $\pi(0) = x$ so that, by definition of $\overrightarrow{\rho}$, $\mathcal{M}, x \models \overrightarrow{\rho}\ \Phi_1[\Phi_2]$. Q.E.D.

Lemma 1. *For all* $n \geq 1$*, formulas* Φ_1*,* Φ_2 *and formula* Ψ_n *as in Definition 4, if* π *is a canonical* $\Phi_1[\Phi_2]$*-path of length* n *over model* $\mathcal{M}$*, then* $\mathcal{M}, \pi(0) \models \Psi_n$*.*

Proof. We proceed by induction on n.
Base case (n=1): if $\texttt{len}\,\pi = 1$, and π is a canonical $\Phi_1[\Phi_2]$-path, then $\mathcal{M}, \pi(i) \models \Phi_1$ for all $i \geq 1$ necessarily, and so also $\mathcal{M}, \pi(1) \models \Phi_1$. This last fact implies, by **Lemma 2**(5) of [2], $\pi(0) \in \overleftarrow{\mathcal{C}}(\pi(1)) \subseteq \overleftarrow{\mathcal{C}}(\llbracket \Phi_1 \rrbracket)$. This means $\mathcal{M}, \pi(0) \models \overleftarrow{\mathcal{N}}\, \Phi_1$ by definition of $\overleftarrow{\mathcal{N}}$ and so, by Definition 4, $\mathcal{M}, \pi(0) \models \Psi_1$.
Induction step: Let π be a canonical $\Phi_1[\Phi_2]$-path of length $n + 1$. Then π', defined as $\pi'(i) = \pi(i + 1)$, is also a canonical $\Phi_1[\Phi_2]$-path and $\texttt{len}\,\pi' = n$. So, by the induction hypothesis, $\mathcal{M}, \pi'(0) \models \Psi_n$, which means $\mathcal{M}, \pi(1) \models \Psi_n$ since $\pi(1) = \pi'(0)$ by definition of π'. Moreover, since $n+1 \geq 2$, path π has at least two elements before the ending loop. This implies that $\mathcal{M}, \pi(1) \models \Phi_2$ as well, since π is a canonical $\Phi_1[\Phi_2]$-path. That is, $\pi(1) \in \llbracket \Phi_2 \wedge \Psi_n \rrbracket$. By **Lemma 2**(5) of [2], we have that $\pi(0) \in \overleftarrow{\mathcal{C}}(\pi(1)) \subseteq \overleftarrow{\mathcal{C}}(\llbracket \Phi_2 \wedge \Psi_n \rrbracket)$. So, we have $\mathcal{M}, \pi(0) \models \overleftarrow{\mathcal{N}}\,(\Phi_2 \wedge \Psi_n)$ by definition of $\overleftarrow{\mathcal{N}}$ and thus $\mathcal{M}, \pi(0) \models \Psi_{n+1}$. Q.E.D.

Lemma 2. *For all $n \geq 1$, formulas Φ_1, Φ_2 and formula Ψ_n as in Definition 4, if $\mathcal{M}, x \models \Psi_n$ then there exists a $\Phi_1[\Phi_2]$-path π such that $\pi(0) = x$ and $\texttt{len}\,\pi \leq n$.*

Proof. By induction on n.
Base case (n=1): $\mathcal{M}, x \models \Psi_1$ if and only if $\mathcal{M}, x \models \overleftarrow{\mathcal{N}}\, \Phi_1$, by Definition 4, if and only if $x \in \overleftarrow{\mathcal{C}}(\llbracket \Phi_1 \rrbracket)$, by definition of $\overleftarrow{\mathcal{N}}$. By definition of $\overleftarrow{\mathcal{C}}$, $x \in \llbracket \Phi_1 \rrbracket$ or there exists $y \in \llbracket \Phi_1 \rrbracket$ such that $(x, y) \in R$, where R is the relation underlying the closure operator[4] $\overrightarrow{\mathcal{C}}$. In the first case, let π be defined as $\pi(j) = x$ for $j \geq 0$. Trivially, π is a $\Phi_1[\Phi_2]$-path, $\pi(0) = x$ and $\texttt{len}\,\pi = 0 \leq 1$. In the second case, let π be defined as $\pi(0) = x$ and $\pi(j) = y$ for $j \geq 1$. This function π is continuous, as shown by Lemma 3 in the Appendix, and so it is a $\Phi_1[\Phi_2]$-path. Moreover, $\pi(0) = x$ and $\texttt{len}\,\pi = 1 \leq 1$.
Induction step: Suppose $\mathcal{M}, x \models \Psi_{n+1}$, that is $\mathcal{M}, x \models \overleftarrow{\mathcal{N}}\,(\Phi_2 \wedge \Psi_n)$, by Definition 4. This means $x \in \overleftarrow{\mathcal{C}}(\llbracket \Phi_2 \wedge \Psi_n \rrbracket)$ by definition of $\overleftarrow{\mathcal{N}}$. By definition of $\overleftarrow{\mathcal{C}}$, $x \in \llbracket \Phi_2 \wedge \Psi_n \rrbracket$ or there exists $y \in \llbracket \Phi_2 \wedge \Psi_n \rrbracket$ such that $(x, y) \in R$. In the first case, we have that $\mathcal{M}, x \models \Psi_n$ and, by the Induction Hypothesis, there exists $\Phi_1[\Phi_2]$-path π such that $\pi(0) = x$ and $\texttt{len}\,\pi \leq n \leq n + 1$. In the second case, by the Induction Hypothesis we know that there exists $\Phi_1[\Phi_2]$-path π' such that $\pi'(0) = y$ and $\texttt{len}\,\pi' \leq n$; moreover, by hypothesis we also know that $\mathcal{M}, y \models \Phi_2$ and that $y \in \overrightarrow{\mathcal{C}}(\{x\})$, since $(x, y) \in R$. We define π as follows: $\pi(0) = x$ and $\pi(i + 1) = \pi'(i)$ for $i = 0 \ldots \texttt{len}\,\pi'$. By Lemma 4 in the Appendix we know that π is continuous; moreover π is a $\Phi_1[\Phi_2]$-path since π' is a $\Phi_1[\Phi_2]$-path, $\pi(1) = \pi'(0) = y$ so $\mathcal{M}, \pi(1) \models \Phi_2$ and since $\texttt{len}\,\pi' \leq n$ we also get $\texttt{len}\,\pi \leq n + 1$. Q.E.D.

[4] Recall that $\mathcal{M} = (X, \overrightarrow{\mathcal{C}}, \mathcal{V})$, where $\overrightarrow{\mathcal{C}}$ (and not $\overleftarrow{\mathcal{C}}$) is the reference closure operator.

4 Finite models

We close these notes by noting that if $\mathcal{M} = (X, \vec{\mathcal{C}}, \mathcal{V})$ is finite, then the length of any canonical path over $\mathcal{M}$ is bounded by the cardinality $|X|$ of X. Moreover, we recall that for each $\Phi_1[\Phi_2]$-path π there exists a canonical $\Phi_1[\Phi_2]$-path π' such that $\pi(0) = \pi'(0)$. Thus, the infinite disjunction in Theorem 1 can be replaced by $\bigvee_{j=1}^{|X|} \Psi_j$:

Corollary 1. *Suppose X is finite. Then, for all $x \in X$ and Ψ_j as in Definition 4 the following holds: $\mathcal{M}, x \models^{\vec{\rho}} \Phi_1[\Phi_2]$ if and only if $\mathcal{M}, x \models \bigvee_{j=1}^{|X|} \Psi_j$.*

Appendix

Lemma 3. *Let $x, y \in X$ with $(x, y) \in R$, where R is the relation underlying $\vec{\mathcal{C}}$. Let furthermore π be defined as $\pi(0) = x$ and $\pi(j) = y$ for $j \geq 1$. Then π is a continuous function.*

Proof. Let $N \subseteq \mathbb{N}$. We have to show that $\pi(\mathcal{C}_{\mathtt{succ}}(N)) \subseteq \vec{\mathcal{C}}\,(\pi(N))$. If $N = \emptyset$ the assert follows trivially. If $N \neq \emptyset$ and $0 \notin N$, then we also have $0 \notin \mathcal{C}_{\mathtt{succ}}(N)$ by definition of $\mathcal{C}_{\mathtt{succ}}$. This implies that $\pi(\mathcal{C}_{\mathtt{succ}}(N)) = \{y\} = \pi(N)$ and then $\vec{\mathcal{C}}\,(\pi(N)) = \vec{\mathcal{C}}\,(\{y\})$. So, we get $\pi(\mathcal{C}_{\mathtt{succ}}(N)) = \{y\} \subseteq \vec{\mathcal{C}}\,(\{y\}) = \vec{\mathcal{C}}\,(\pi(N))$ where we used that $A \subseteq \vec{\mathcal{C}}\,(A)$ for all $A \subseteq X$. Finally, if $0 \in N$, we have that $\{0, 1\} \subseteq \mathcal{C}_{\mathtt{succ}}(N)$ so that $\pi(\mathcal{C}_{\mathtt{succ}}(N)) = \{x, y\}$. Moreover, since $x = \pi(0) \in \pi(N)$, noting that $y \in \vec{\mathcal{C}}\,(\{x\})$ since $(x, y) \in R$, we get that $\{x, y\} \subseteq \vec{\mathcal{C}}\,(\pi(N))$. So we get $\pi(\mathcal{C}_{\mathtt{succ}}(N)) = \{x, y\} \subseteq \vec{\mathcal{C}}\,(\pi(N))$. Q.E.D.

Lemma 4. *Let $x, y \in X$ with $(x, y) \in R$. Let π' be a bounded path over $\mathcal{M}$, with $\pi'(0) = y$ and define π as $\pi(0) = x$ and $\pi(i + 1) = \pi'(i)$ for $i = 0 \dots \mathtt{len}\,\pi'$. Then π is a continuous function.*

Proof. Let N be an arbitrary subset of $\mathbb{N}$. We have to show that $\pi(\mathcal{C}_{\mathtt{succ}}(N)) \subseteq \vec{\mathcal{C}}\,(\pi(N))$. We first of all note that for all $N \subseteq \mathbb{N} \setminus \{0\}$ we have:

i. $\mathcal{C}_{\mathtt{succ}}(\{i - 1 \,|\, i \in N\}) = \{i - 1 \,|\, i \in N\} \cup N$ and
ii. $\{i - 1 \,|\, i \in \{j \,|\, j - 1 \in N\}\} = N$.

We proceed separately for the case in which $0 \notin N$ and $0 \in N$.
case $0 \notin N$:

$$
\begin{aligned}
& \pi(\mathcal{C}_{\mathtt{succ}}(N)) \\
= \quad & [\,\text{Def. of } \mathcal{C}_{\mathtt{succ}}\,] \\
& \pi(N \cup \{i \,|\, i - 1 \in N\}) \\
= \quad & [\,\pi(A \cup B) = \pi(A) \cup \pi(B) \text{ for all } A, B \subseteq X\,] \\
& \pi(N) \cup \pi(\{i \,|\, i - 1 \in N\})
\end{aligned}
$$

$=$ [Def. of π in terms of π']

$\pi'(\{i-1 \mid i \in N\}) \cup \pi'(\{i-1 \mid i \in \{j \mid j-1 \in N\}\})$

$=$ [Point (ii) above]

$\pi'(\{i-1 \mid i \in N\}) \cup \pi'(N)$

$=$ [$\pi'(A) \cup \pi'(B) = \pi'(A \cup B)$ for all $A, B \subseteq X$]

$\pi'(\{i-1 \mid i \in N\} \cup N)$

$=$ [Point (i) above]

$\pi'(\mathcal{C}_{\mathsf{succ}}(\{i-1 \mid i \in N\}))$

$\subseteq$ [π' continuous by hypothesis]

$\overrightarrow{\mathcal{C}}(\pi'(\{i-1 \mid i \in N\}))$

$=$ [Def. of π in terms of π']

$\overrightarrow{\mathcal{C}}(\pi(N))$

case $0 \notin N$:
First of all note that $\pi(\overrightarrow{\mathcal{C}}_{\mathsf{succ}}(\{0\})) \subseteq \overrightarrow{\mathcal{C}}(\pi(\{0\}))$. In fact $\pi(\overrightarrow{\mathcal{C}}_{\mathsf{succ}}(\{0\})) = \pi(\{0,1\}) = \{\pi(0), \pi(1)\}$ and by hypothesis $\pi(0) = x$ and $\pi(1) = y \in \overrightarrow{\mathcal{C}}(\{x\})$, since $(x, y) \in R$, and $\overrightarrow{\mathcal{C}}(\{x\}) = \overrightarrow{\mathcal{C}}(\{\pi(0)\})$; so $\pi(\overrightarrow{\mathcal{C}}_{\mathsf{succ}}(\{0\})) = \{\pi(0), \pi(1)\} \subseteq \overrightarrow{\mathcal{C}}(\{\pi(0)\})$. We proceed with the following derivation:

$\pi(\overrightarrow{\mathcal{C}}_{\mathsf{succ}}(N))$

$=$ [Def. of $\overrightarrow{\mathcal{C}}_{\mathsf{succ}}$]

$\pi(N \cup \{i \mid i-1 \in N\})$

$=$ [$\pi(A \cup B) = \pi(A) \cup \pi(B)$ for all $A, B \subseteq X$]

$\pi(N) \cup \pi(\{i \mid i-1 \in N\})$

$=$ [Let $\hat{N} = N \setminus \{0\}$]

$\pi(\{0\} \cup \hat{N}) \cup \pi(\{1\} \cup \{i \mid i-1 \in \hat{N}\})$

$=$ [$\pi(A \cup B) = \pi(A) \cup \pi(B)$ for all $A, B \subseteq X$]

$\pi(\{0\}) \cup \pi(\hat{N}) \cup \pi(\{1\}) \cup \pi(\{i \mid i-1 \in \hat{N}\})$

$=$ [Rearranging]

$\pi(\hat{N}) \cup \pi(\{i \mid i-1 \in \hat{N}\}) \cup \pi(\{0,1\})$

$=$ [Def. of π in terms of π' and Point (ii) above (as before, but on $\hat{N}$)]

$$
\begin{aligned}
& \pi'(\{i-1 \,|\, i \in \hat{N}\}) \cup \pi'(\hat{N}) \cup \pi(\{0,1\}) \\
= \quad & [\, \pi(A) \cup \pi(B) = \pi(A \cup B) \text{ for all } A, B \subseteq X \,] \\
& \pi'(\{i-1 \,|\, i \in \hat{N}\} \cup \hat{N}) \cup \pi(\{0,1\}) \\
= \quad & [\, \text{Point (i) above; Def. of } \mathcal{C}_{\mathsf{succ}}(\{0\}) \,] \\
& \pi'(\mathcal{C}_{\mathsf{succ}}(\{i-1 \,|\, i \in \hat{N}\})) \cup \pi(\mathcal{C}_{\mathsf{succ}}(\{0\})) \\
\subseteq \quad & [\, \pi' \text{ is continuous by hypothesis} \,] \\
& \vec{\mathcal{C}}\,(\pi'(\{i-1 \,|\, i \in \hat{N}\})) \cup \pi(\mathcal{C}_{\mathsf{succ}}(\{0\})) \\
= \quad & [\, \text{Def. of } \pi \text{ in terms of } \pi' \,] \\
& \vec{\mathcal{C}}\,(\pi(\hat{N})) \cup \pi(\mathcal{C}_{\mathsf{succ}}(\{0\})) \\
\subseteq \quad & [\, \pi(\mathcal{C}_{\mathsf{succ}}(\{0\})) \subseteq \vec{\mathcal{C}}\,(\pi(\{0\})) \text{: see above} \,] \\
& \vec{\mathcal{C}}\,(\pi(\hat{N})) \cup \vec{\mathcal{C}}\,(\pi(\{0\})) \\
= \quad & [\, \vec{\mathcal{C}}\,(A) \cup \vec{\mathcal{C}}\,(B) = \vec{\mathcal{C}}\,(A \cup B) \text{ for all } A, B \subseteq X \,] \\
& \vec{\mathcal{C}}\,(\pi(\hat{N}) \cup \pi(\{0\})) \\
= \quad & [\, \pi(A) \cup \pi(B) = \pi(A \cup B) \text{ for all } A, B \subseteq X \,] \\
& \vec{\mathcal{C}}\,(\pi(\hat{N} \cup \{0\})) \\
= \quad & [\, N = \hat{N} \cup \{0\} \text{ by definition} \,] \\
& \vec{\mathcal{C}}\,(\pi(N))
\end{aligned}
$$

Q.E.D.

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